How hot is radiation?

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A self-consistent approach to defining nonequilibrium radiation temperature is discussed using the distribution of the energy over microstates. We begin rigorously with ensembles of Hilbert spaces and end with practical examples based mainly on the far from equilibrium radiation of lasers. We show that very high, but not infinite, laser radiation temperatures depend on intensity and frequency. Heuristic definitions of temperature derived from a misapplication of equilibrium arguments are shown to be incorrect. More general conditions for the validity of nonequilibrium temperatures also are established. © *2003 American Association of Physics Teachers.*
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I. INTRODUCTION

The standard definitions of intensive thermodynamic parameters, such as temperature, seem to require the system in question to be in thermodynamic equilibrium. In this paper, we explore the entropy and temperature of radiation out of equilibrium and show that, within limited restrictions that do not require equilibrium, the radiation temperature is well-defined and distinct from any associated matter temperature and from plausible but incorrectly applied equilibrium definitions. We use laser radiation as our major example.

Laser radiation is a fascinating example of a highly organized quantum system of quasi-coherent bosons.[1–3] A laser beam is supported by external pumping, which keeps the beam far from thermodynamic equilibrium. The laser shares this feature with other steady-state systems that are kept from equilibrating by external constraints. We find that the temperature of laser radiation far exceeds the temperatures of the laser cavity and the lasing atomic transition. Misidentifying the radiation with the matter temperature leads to erroneous estimates of the laser radiation temperature which are as much as ten orders of magnitude too small.

The Gibbs–Duhem relation for radiation, $S\,dT - V\,dP = 0$, implies that the two intensive thermodynamic parameters, pressure P (conjugate to volume) and temperature T (conjugate to energy), reduce to one independent intensive parameter, which is usually identified as T. This feature of radiation thermodynamics, like the photon's zero mass and lack of rest frame, makes radiation thermodynamics much simpler than that of matter, which has conserved particle numbers and nonzero chemical potentials.[4,5] It also makes generalizing intensive thermodynamic parameters out of equilibrium much easier. Thus radiation is a natural context in which to introduce nonequilibrium temperature.

A properly defined nonequilibrium temperature has physical meaning. It occurs in the *entropy production rate* Σ, an important measure of both how far a system is from equilibrium and how fast it is approaching equilibrium.[6] The generic form of Σ is rooted in the equilibrium expression for the entropy differential, $dS = dQ/T$, where dQ is the change in the energy of the system while holding the volume and particle number constant. The form $\Sigma \sim J_Q(12)(1/T_1 - 1/T_2)$ expresses the entropy produced by two subsystems (1,2) at temperatures T_1, T_2 as they exchange a heat flux $J_Q(12)$. Subsystem temperatures occur naturally in expressions for entropy production. Σ is positive semidefinite and vanishes if and only if $T_1 = T_2$, the condition for thermal equilibrium. The heat flux $J_Q(12)$ vanishes in this case as well.

The difference $1/T_1 - 1/T_2$ of the inverse temperatures is a measure of how far out of equilibrium the two subsystems are with respect to one another. The heat flux $J_Q(12)$ is a measure of how fast the subsystems are approaching equilibrium with one another, assuming no external pumping of the system. [With external pumping, $J_Q(12)$ is a measure of how much power has to be injected into the system to keep it from equilibrating.] The product of these two quantities combines the two measures into a single quantity characteristic of a nonequilibrium process.

The books by Reichl[5] and De Groot and Mazur[6] explain the significance and role of entropy production in nonequilibrium matter systems. Section VII and Refs. 4 and 7 explore entropy production in radiation and radiation-matter systems.

II. MISAPPLICATIONS OF TEMPERATURE OUT OF EQUILIBRIUM

Many expressions for various physical quantities have units of temperature. These expressions frequently are defined as the temperatures that equilibrium systems would have if the energy, entropy, number, etc., were all rearranged in some particular way. There are many ways to rearrange a system into an equilibrium state. Consequently, many definitions of these pseudo-temperatures are possible, but none represent the temperature of the actual system, and the application of temperature to nonequilibrium systems is ambiguous.[8]

Consider a helium–neon laser with the transition line $\lambda_0 = 632.8$ nm. A heuristic but incorrect equipartition argument that sets the mean photon energy equal to $k_B T$ would associate this transition with a temperature $k_B T_0 = hc/\lambda_0$ or T_0

$\sim 2\times 10^4$ K. The quantity T_0 is the temperature that a blackbody would have if the distribution of energy among the photons were contrived to have an average photon energy corresponding to the photons in question. But the rest of the distribution is far from a blackbody for any laser. T_0 refers to a photon gas with a different distribution, energy, and entropy. So T_0 has no direct physical meaning for laser radiation, which is far from equilibrium with itself and with its matter source.

Laser radiation often is idealized as having infinite temperature, as lasers are interpreted as a source of pure work, although ultimately this interpretation is unsatisfactory. It would put a powerful x-ray laser on an equal footing with a pocket red diode laser pointer, powered by watch batteries. Is there no difference in their temperatures? There seems to be none if both lasers are described by infinite temperature.

Other possible pseudo-temperatures include the temperatures that a beam would have if the same energy or the same entropy were arranged differently, for example, in a blackbody distribution. Such temperatures do not reflect the actual distribution of energy, entropy, and photon number in the beam. Different as these definitions are from each other, they would all agree if the laser radiation were forced into thermodynamic equilibrium.

There are other nonequilibrium temperature definitions that reflect the actual energy and entropy distributions. For example, objects in the laboratory surrounding the laser have meaningful local temperatures. The material around a He–Ne laser is at room temperature $T_{\text{room}} \simeq 300$ K, and the He–Ne gas is at $T_{\text{gas}} \simeq 400$ K.

The inverted populations N_2 and N_1 of the upper and lower laser energy levels E_2 and E_1 are associated with a temperature T_{21} through a formal Boltzmann distribution,

$$N_2/N_1 = \exp[-(E_2-E_1)/k_B T_{21}], \tag{1}$$

which does not hold for all levels. Such a pseudo-temperature can be defined for any two levels. In this case, the definition and population inversion imply a negative value for T_{21}. But such a definition lacks a thermodynamic justification.

Although an inverted atomic population is essential to lasing action, there is no reason to attribute the temperature T_{21} to the radiation field, a separate entity with its own thermodynamics. In Secs. IV and V we will identify the natural radiation temperature that represents its distinctive nature, and that points to a more general definition in terms of an energy distributed over states. Temperature emerges not as a proxy for energy, but instead as a measure of how the energy is organized among the various microstates. It possesses a rigorous definition and plays a natural role in nonequilibrium systems that go beyond *ad hoc* heuristic estimates.[2]

III. DYNAMICS OF THE RADIATION FIELD

We analyze the radiation field in a box and decompose it into plane wave modes.[1,9] Each field mode of wave vector $\mathbf{k}$ and given polarization fills space. (We ignore polarization in this paper for simplicity, which does not change the generality of the argument.) The fundamental mode variable is its complex amplitude $\alpha_{\mathbf{k}} = a_{\mathbf{k}} e^{i\phi_{\mathbf{k}}}$, in terms of its modulus $a_{\mathbf{k}}$ and phase $\phi_{\mathbf{k}}$.

Each mode is a linear harmonic oscillator and can be mapped onto a quantum harmonic oscillator in terms of lowering (raising) operators $\hat{a}_{\mathbf{k}}$ ($\hat{a}_{\mathbf{k}}^\dagger$), satisfying $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}}^\dagger] = \hat{1}_{\mathbf{k}}$, for each mode. The state of a mode lives in an infinite-dimensional Hilbert space. The occupation number basis $|n\rangle$ forms a complete orthonormal set of number operator eigenvectors corresponding to different photon numbers: $\hat{n}_{\mathbf{k}}|n_{\mathbf{k}}\rangle = \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}|n_{\mathbf{k}}\rangle = n_{\mathbf{k}}|n_{\mathbf{k}}\rangle$, and $\Sigma_n |n\rangle\langle n| = \hat{1}$. The field Hamiltonian $\hat{H} = \Sigma_{\mathbf{k}} h\nu(\hat{n}_{\mathbf{k}} + 1/2)$, with $\nu = ck$. The state of the whole field is the direct product of all its mode states and lives in a field Hilbert space (Fock space).

An alternative basis is the set of coherent states $|\alpha_{\mathbf{k}}\rangle$, eigenstates of the lowering operator: $\hat{a}_{\mathbf{k}}|\alpha_{\mathbf{k}}\rangle = \alpha_{\mathbf{k}}|\alpha_{\mathbf{k}}\rangle$, defined so that the occupation number expectation $N_{\mathbf{k}} = \langle \alpha_{\mathbf{k}}|\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}|\alpha_{\mathbf{k}}\rangle = |\alpha_{\mathbf{k}}|^2 = a_{\mathbf{k}}^2$. Coherent states satisfy

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \hat{1}, \quad \langle\beta|\alpha\rangle = e^{-(1/2)|\alpha-\beta|^2 + i\,\mathrm{Im}(\beta^*\alpha)},$$
$$\langle n|\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2}. \tag{2}$$

The state $|\alpha_{\mathbf{k}}\rangle$ is the quantum analogue of a single classical mode $\mathbf{k}$ with amplitude $\alpha_{\mathbf{k}}$. Physical results can be obtained with either the coherent state basis or the photon occupation number basis $|n\rangle$.[9]

Problem 1: Use the definition of $\langle n|\alpha\rangle$ to derive the above identity: the result for $\langle\beta|\alpha\rangle$ and the integral identity for the coherent states $|\alpha\rangle$. Also show directly, by expanding $|\alpha\rangle$ in $|n\rangle$, that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$.

Thermodynamics requires a statistical description with an ensemble of many Fock spaces. Each mode has an ensemble of amplitudes and phases, and the ensemble of the whole field is the direct product of the mode ensembles.[5]

Very high temperatures are to be expected in low entropy, high energy bosonic systems far from thermodynamic equilibrium such as the laser. Such high temperatures are natural for bosons, because they lack an exclusion principle. Ideally, a multiparticle bosonic system could be driven to zero entropy by putting all particles in one single-particle state, without changing the system's total energy. As shown in Sec. IV C, if the total energy remains fixed, the temperature of that state would be infinite.

IV. NONEQUILIBRIUM THERMODYNAMICS

A system's statistical ensemble defines the system's entropy. Defining the temperature of a system or subsystem necessitates the restriction to cases where its entropy and energy have a functional relationship. A physically meaningful definition of temperature should reflect the system's actual state and properly reduce to equilibrium temperature.

A. The entropy in general

To define the entropy, we introduce an ensemble of a large number M of copies of a system assumed to be made up of discrete, countable ''things.'' Physically, these things are the system's fundamental degrees of freedom. Each copy is microscopically different and distinguishable from the others.

Consider the entropy of two system copies. Let W_1 be the overall number of ways that the first group of things can be arranged, and W_2 the corresponding number for the second group of things. The overall number of ways W that the two groups can be arranged is the product $W = W_1 \times W_2$. For the

entropy, we want a function of W that matches the usual definition for the special case of equilibrium. Therefore, this function of W must depend on the two combined subsystems by adding the entropies of the separate subsystems. Such a function must be a linear function of the logarithm of W. The definition $S=k_B \ln W$, up to a multiplicative factor and an irrelevant additive constant, is the only function that does so. Note that entropy should be additive, but need not scale in a simple way with the size of the system. For a nonequilibrium system, although subsystems contribute additively to the whole, the subsystems do not have to be homogeneous and contribute to the whole in a simple scaling fashion.

Each system copy has the same internal probabilities p_σ for being in any particular state σ. The number of system copies in state σ is $Mp_\sigma=m_\sigma$. Suppose that the ensemble has m_1 systems in state σ_1, m_2 systems in state σ_2, etc. The number of ways this ensemble can be created from the system copies is

$$W_M=M!/[m_1!m_2!\cdots m_\sigma!\cdots], \tag{3}$$

and the entropy for each member of the ensemble is $S=k_B \ln(W_M)/M$. For M large, Stirling's approximation yields

$$MS=k_B\left[M \ln M-\sum_\sigma (m_\sigma \ln m_\sigma)\right]. \tag{4}$$

(Stirling's approximation is $n!\sim n^n e^{-n}\sqrt{2\pi n}$ for n large. We ignore additive constants in the entropy, as these are independent of the system's thermodynamic state.)

Because $m_\sigma=Mp_\sigma$, we have

$$\begin{aligned} S&=k_B\left[\ln M-\sum_\sigma (p_\sigma \ln M+p_\sigma \ln p_\sigma)\right] \\ &=k_B\left[\left(1-\sum_\sigma p_\sigma\right)\ln M-\sum_\sigma p_\sigma \ln p_\sigma\right] \\ &=-k_B\sum_\sigma p_\sigma \ln p_\sigma, \end{aligned} \tag{5}$$

where the last line follows because $\Sigma_\sigma p_\sigma=1$ and the ensemble's system copies are independent.

A thermodynamic system's statistical ensemble is equivalent to its normalized, Hermitian density operator $\hat{\rho}$: $\mathrm{Tr}(\hat{\rho})=\Sigma_\sigma p_\sigma=1$. The p_σ are the eigenvalues of $\hat{\rho}$, with eigenvectors $|\sigma\rangle$. Any observable $\mathcal{O}$ has a statistical average

$$\langle\mathcal{O}\rangle=\mathrm{Tr}(\hat{\rho}\hat{\mathcal{O}})=\sum_\sigma p_\sigma\langle\sigma|\hat{\mathcal{O}}|\sigma\rangle. \tag{6}$$

The average of the operator $-k_B \ln\hat{\rho}$ is the entropy, $S=-k_B\mathrm{Tr}(\hat{\rho}\ln\hat{\rho})$. The mean energy $E=\mathrm{Tr}(\hat{\rho}\hat{H})$ is the average of the Hamiltonian $\hat{H}$.[5]

B. The validity of nonequilibrium temperature

Out of equilibrium, the entropy S lacks a clear functional dependence on the total energy E, and the definition of T becomes ambiguous. However, an important generalization of T is still possible, with certain restrictions. The crucial requirement is that the whole system decompose into subsystems whose entropies are functions of each subsystem's energy only. That is, the nonequilibrium system must decompose into subsystems each with its own equilibrium. Restriction to full equilibrium is unnecessary.

We write the complete system Hamiltonian as $\hat{H}=\Sigma_p \hat{H}_p+\Sigma_{q>p}\hat{H}_{pq}$, where the first sum is over separate subsystems and the second is over all interactions between subsystems. The full $\hat{\rho}$ evolves via the quantum Liouville equation $i\hbar d\hat{\rho}/dt=[\hat{H},\hat{\rho}]$. For a particular subsystem p to have well-defined thermodynamics, the necessary restrictions are the following.

(1) The whole system's density operator $\hat{\rho}$ must be factorizable into independent subsystems: $\hat{\rho}=\otimes_p\hat{\rho}_p$. This factorization implies that the total system entropy S is a sum over subsystems: $S=\Sigma_p S_p$.

(2) The commutator $[\hat{H}_p,\hat{\rho}_p]$ must be negligible or zero. Then $\hat{\rho}_p$ and $\hat{H}_p$ are simultaneously diagonalizable, and a functional relationship $\hat{\rho}_p=\hat{\rho}_p(\hat{H}_p)$ is possible. The subsystem's ensemble probabilities are $p_{p_\sigma}=\langle\sigma|\hat{\rho}_p(\hat{H}_p)|\sigma\rangle=p_{p_\sigma}(E_{p_\sigma})$. That is, each eigenvalue p_{p_σ} of $\hat{\rho}_p$ is a function of the corresponding eigenvalue E_{p_σ} *only*. It is not a function of the other energy eigenvalues E_{p_μ} of subsystem p or of the $\{E_{q_\mu}\}$ of the other subsystems q. (If the interactions $\hat{H}_{pq}$ between subsystems are negligible, then the total system energy E is a sum over the E_p's and each subsystem is in a stationary state. But this restriction is not necessary. For example, laser radiation modes in the laser cavity are strongly coupled to external pumping.)

These special conditions allow a functional relationship between ensemble averages of entropy and energy for each subsystem that does not depend on the specific probability distribution $\{p_\sigma\}$. That is, $S_p=S_p(E_p)$, which leads to a natural temperature of subsystem p:

$$\frac{1}{T_p}=\frac{\partial S_p}{\partial E_p}, \tag{7}$$

where T_p is a rate of change between extensive quantities, as it normally is in thermodynamics. Such a relationship exists for photons if the subsystems are chosen over small enough ranges of energy. We note that (i) the functional relationship $S_p=S_p(E_p)$ is an equation of state. (ii) If S_p is a function of E_q, for $q\neq p$, the temperature T_p is still defined, but is not a function of subsystem p alone. (iii) If many subsystems are in contact with one another, their temperatures represent a basis for their complete thermal equilibrium. For each pair of subsystems (p,q), $T_p=T_q=T$, where T is the single temperature of the whole system: $1/T=\partial S/\partial E$.

A familiar example of such a subsystem temperature occurs in the local thermodynamic equilibrium of the radiation field typical of stellar interiors.[7,10] In this case, periodic boundary conditions are applied in small local boxes at positions $\mathbf{r}$. A different field for each Fourier mode $\mathbf{k}$, ensemble, and temperature $T(\mathbf{k})$ can be defined in each box. (Very long wavelength modes that do not fit into the small local boxes must be ignored.) Thus the subsystem is a particular mode $\mathbf{k}$ in a box at $\mathbf{r}$, and T_p becomes a photon phase space temperature $T(\mathbf{r},\mathbf{k})$. At a point $\mathbf{r}$ in space, this brightness temperature of the photons is a function of mode frequency $\nu=ck$ and direction $\hat{\mathbf{k}}$ and is conventionally written

as $T_\nu(\hat{\mathbf{k}})$. In Secs. V and VI, we show how this temperature emerges naturally as a genuine thermodynamic one.

C. Pure states and infinite temperature

A pure state has zero entropy. One of the p_σ (say p_τ) is unity, while the others vanish, and so $S=0$. There are many ways to realize a pure state with an idealized laser. The two simplest are a pure coherent state $\hat{\rho}=|\alpha\rangle\langle\alpha|$ or pure occupation number state $\hat{\rho}=|n\rangle\langle n|$.

Problem 2: Generally, $\hat{\rho}$ is a mixture of projection operators over different states; for example, in the number basis, $\hat{\rho}=\Sigma p_{nm}|n\rangle\langle m|$. But the ensemble of a pure state $|\psi\rangle$ is $\hat{\rho}=|\psi\rangle\langle\psi|$, a single projection operator, not a mixture. Prove that the eigenvalues of a pure ensemble can only be one or zero. Hint: A normalized projection operator $\hat{\Pi}=|\psi\rangle\langle\psi|$, with $\langle\psi|\psi\rangle=1$, satisfies $\hat{\Pi}^2=\hat{\Pi}$. Then infer that the eigenvalues p of a pure $\hat{\rho}$ satisfy the formally identical equation, $p^2=p$.

We define a single mode's ensemble by $\{p_\sigma\}$. The ensemble average energy is given by $E=\Sigma_\sigma p_\sigma E_\sigma$, where $E_\sigma=\langle\sigma|\hat{H}|\sigma\rangle$. The subsystem temperature is defined by the derivative,

$$\frac{1}{k_B T}=-\frac{\Sigma_\sigma \ln p_\sigma\, dp_\sigma}{\Sigma_\sigma E_\sigma\, dp_\sigma}, \tag{8}$$

where $\Sigma_\sigma dp_\sigma=0$.

This definition of subsystem temperature is a simple restatement of the thermal equilibrium of subsystems across ensemble members. This argument does not require all members of the ensemble to participate, but the probabilities are assumed to be normalized over the active members of the ensemble, with the rest are ignored. Equation (8) thus provides a legitimate equilibrated temperature of only those ensemble members indicated by the summation.

Consider two cases based on a sequence of equilibrated subsystem probability distributions with the temperature defined by Eq. (8).

(1) In equilibrium across all ensemble members, the p_σ are very small, as the probability is spread over the entire ensemble. As the mean ensemble energy E falls, while maintaining equilibrium, the average is dominated by the lowest-lying value of the set $\{E_\sigma\}$ as the lower and lower energy states are occupied. Then $\{p_\sigma\}$ becomes dominated by one $p_\tau\to 1$, while the others vanish. But in most systems, the denominator in Eq. (8) vanishes faster than the slowly changing logarithm in the numerator. So in equilibrium, $T\to 0$ as E and $S\to 0$. The classic example is the Bose–Einstein condensate.[5]

(2) In the nonequilibrium case of interest in this paper, the mean ensemble energy E is fixed. Now redistribute this energy and the ensemble probability into ensembles restricted to successively smaller subspaces of the Hilbert space. Equilibrate the energy on these successively smaller ensembles. Equation (8) still holds on these smaller ensembles, but E and the $\{p_\sigma\}$ are distributed over fewer states. One p_τ approaches unity and the others vanish. The number of ensemble members declines toward one, the pure state limit, yielding, in the limit of one possible subsystem in the ensemble, the limit,

$$\frac{1}{k_B T}\sim -\frac{\ln p_\tau}{E_\tau}\to 0. \tag{9}$$

So $T\to\infty$ for a pure state if the total energy is fixed while the entropy vanishes.

These results show how temperature is related not only to the ensemble energies E_σ, but to the ensemble distribution $\{p_\sigma\}$ as well.

The radiation field of a real laser is not in a pure state. Each mode has a statistical ensemble of amplitudes or occupation numbers, with an associated entropy. Thus, whether the field is given a classical or quantum description, a real laser does not have infinite temperature, although laser temperatures are often idealized as infinite.

V. RADIATION ENTROPY AND TEMPERATURE

From this point, we will work with individual photon modes $\mathbf{k}$ and drop mode labels where they are not needed. Statistical ensembles of radiation can be described by density operators $\hat{\rho}$.[1,3,5] Although a real laser is not in a coherent state, the coherent state basis is exceptionally useful, because it connects the radiation mode description in terms of amplitude and phase with photon number and energy.

In principle, we could calculate the entropy S and other thermodynamic functions from $\hat{\rho}$ and find T. But S is difficult to calculate for arbitrary $\hat{\rho}$. We consider only the special case where the radiation field's statistical ensemble is independent of the phase ϕ. For most lasers, the phase is fully randomized. The mode energy $E=h\nu N$ is always phase-independent, depending only on the ensemble's photon number expectation $N=\mathrm{Tr}(\hat{a}^\dagger\hat{a})$. Thus the phase does not affect S or T in this restricted case. This restriction simplifies $S(N)$ to a general nonequilibrium form which we can obtain by a simple counting argument appropriate for bosons.[11]

A. Coherent state basis: Random field phase

The density operator can be expanded over coherent states, $\hat{\rho}=\int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|$, and then projected on to the $|n\rangle$ basis, $p_{nm}=\langle n|\hat{\rho}|m\rangle$. The normalization is fixed, $\mathrm{Tr}(\hat{\rho})=\int d^2\alpha\ P(\alpha)=\Sigma_n p_{nn}=1$.[12] The entropy is given by $S=-k_B\mathrm{Tr}(\hat{\rho}\ln\hat{\rho})=-k_B\Sigma_n\langle n|\hat{\rho}\ln\hat{\rho}|n\rangle$. We restrict the distribution function $P(\alpha)$ to be independent of phase ϕ and depend only on the modulus a. The matrix p_{nm} then becomes diagonal:

$$\begin{aligned} p_n\equiv p_{nm}&=\int d^2\alpha\langle n|\alpha\rangle\langle\alpha|m\rangle P(\alpha)\\ &=2\pi\delta_{nm}\int_0^\infty da\,\frac{a^{2n+1}}{n!}e^{-a^2}P(a). \end{aligned} \tag{10}$$

The entropy simplifies to $S=-k_B\Sigma_n p_n\ln p_n$. The eigenstates $|\sigma\rangle$ of $\hat{\rho}$ are the occupation number states $|n\rangle$. The distribution $P(\alpha)$ is phase-independent if the phase distribution for the amplitude α is random over the interval $\phi\in[0,2\pi)$.

The restriction to random phase and the result that the mode state can be characterized by counting the photon number alone validates the phase space approach to photon thermodynamics. For each mode of the field labeled by a wave number $\mathbf{k}$, the mode state is determined solely by its mean photon occupation number, $N_{\mathbf{k}}=\Sigma_n p_{\mathbf{k}_n} n_{\mathbf{k}}$.[13,14]

B. Phase space reduction: Counting photons

Making the field's statistical ensemble phase-independent allows us to find $S(N)$ by simply counting states for identical bosons, treating the field quanta as particles. In this case, all of the entropy is due to the randomness of the phase. The reasoning of this section parallels that of Sec. IV A.

Consider N identical things and G identical possible places to put them. Imagine not knowing where the N things are among the G places. Then there are $(N+G-1)!/N!(G-1)!=W$ ways of arranging the N things among the G places. Now introduce an ensemble of M system copies, with each copy labeled by σ. For systems such as lasers with large numbers of photons, we can assume that G and $N \gg 1$ and use Stirling's approximation. G and N depend on the system copy σ in the ensemble. The entropy is then:

$$\begin{aligned} MS &= k_B \sum_\sigma \left[-G_\sigma \ln G_\sigma + (G_\sigma + N_\sigma)\ln(G_\sigma + N_\sigma) \right. \\ &\quad \left. - N_\sigma \ln N_\sigma \right] \\ &= k_B \sum_\sigma G_\sigma [(1+n)\ln(1+n) - n \ln n], \end{aligned} \tag{11}$$

where $n_\sigma = N_\sigma / G_\sigma$ is the mean occupation number in the ensemble and approaches the σ-independent limit $n = N/G$, with $G = \Sigma_\sigma G_\sigma / M$, as $M \to \infty$. Because $h\nu/T = \partial S/\partial N$ and $N = Gn$,

$$k_B T = \frac{h\nu}{\ln(1+1/n)}. \tag{12}$$

Problem 3: Combine Eqs. (11) and (12) to express S in terms of T.

To connect the result (12) to radiation observables, identify the G "places" with the Fourier space of modes $\mathbf{k}$ and use the general relation of mean occupation number n to the specific intensity $I_\nu(\hat{\mathbf{k}})$: $n = c^2 I_\nu(\hat{\mathbf{k}})/h\nu^3$, where $I_\nu(\hat{\mathbf{k}})$ is the radiation energy/area/time/frequency/solid angle.[10,15] The resulting expression for the temperature,

$$k_B T_\nu(\hat{\mathbf{k}}) = \frac{h\nu}{\ln[1 + h\nu^3/c^2 I_\nu(\hat{\mathbf{k}})]}, \tag{13}$$

can also be derived from the Planck blackbody expressions for I_ν by solving for T and *defining* the resulting temperature in terms of I_ν. The value of I_ν can then be arbitrary.[4] The radiation temperature varies as a function of beam direction $\hat{\mathbf{k}}$ as well as frequency ν. Except at zero frequency, T_ν is zero only for vanishing n or I_ν (photon vacuum).

C. Temperature as a Lagrange multiplier

Instead of Eq. (8) with its assumptions about probabilities, the temperature can be considered as a Lagrange multiplier for holding the ensemble mean energy fixed while maximizing the entropy. This procedure can be extended to subsystems, such as a single radiation mode or a collection of modes constituting a Fourier subspace K, a part of the whole mode space. The conclusions of the previous sections are confirmed in a different way, as long as the mode phases are random.

If we express Eq. (11) in Fourier space, the real space volume density s of entropy contributed by photons of wave number $\mathbf{k}$ and frequency $\nu = ck$ in the Fourier subspace K becomes:

$$s = \int_K k_B [(1+n)\ln(1+n) - n \ln n] \frac{d^3k}{(2\pi)^3}, \tag{14}$$

where n is the mean occupation number. The real space volume density e of energy in K is

$$e = \int_K nh\nu \frac{d^3k}{(2\pi)^3}. \tag{15}$$

The maximum entropy at fixed energy is given by

$$0 = \delta(s + \beta e) = \int_K \delta \left[\ln\left(\frac{1+n}{n}\right) - \beta h\nu \right] \frac{d^3k}{(2\pi)^3}, \tag{16}$$

from which we conclude that

$$n = \frac{1}{e^{\beta h\nu} - 1}. \tag{17}$$

We find an apparent blackbody distribution for the specific intensity:

$$I_\nu = \frac{h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1}, \tag{18}$$

and identify the Lagrange multiplier $\beta = 1/k_B T$. We define the specific intensity of the entropy radiation J_ν by photon counting, the same way that the specific energy intensity I_ν is defined.[4] Then we find another version of Eq. (7):

$$\frac{1}{T_\nu(\hat{\mathbf{k}})} = \frac{\partial J_\nu(\hat{\mathbf{k}})}{\partial I_\nu(\hat{\mathbf{k}})}. \tag{19}$$

If K were all of Fourier space, this distribution would be an equilibrium one with a single temperature T. But we have not restricted K. If K is any part of Fourier space, the distribution is a blackbody distribution, but truncated to only its domain of Fourier subspace. This subspace is complete with its own temperature $k_B T[K] = \delta e[K]/\delta s[K]$, even though the whole distribution is not present. The simplicity of radiation thermodynamics converts the functions $s = s(n[K])$ and $e = e(n[K])$ into an implicit function $s(e)$ over K, matching the more general argument of Sec. IV B. Therefore each K forms a distinct thermodynamic system with a temperature given by Eq. (13) or obtainable by inverting Eq. (18).

This argument can be extended to a K of zero volume. By squeezing a fixed radiation energy into a vanishingly small frequency interval $\Delta\nu$ and beam solid angle cone $\Delta\Omega$, we recover the infinite temperature of a pure state for one mode $\mathbf{k}$, with $\nu = ck$, as we now show.

Consider a fixed arbitrary specific intensity I_ν distribution and break it into two parts, a blackbody function B_ν of constant temperature T, plus a finite deviation distribution $D_\nu(\hat{\mathbf{k}})$, which may depend on the direction $\hat{\mathbf{k}}$. Thus $I_\nu = B_\nu + D_\nu$. The energy density u within $(\Delta\nu, \Delta\Omega)$ is given by

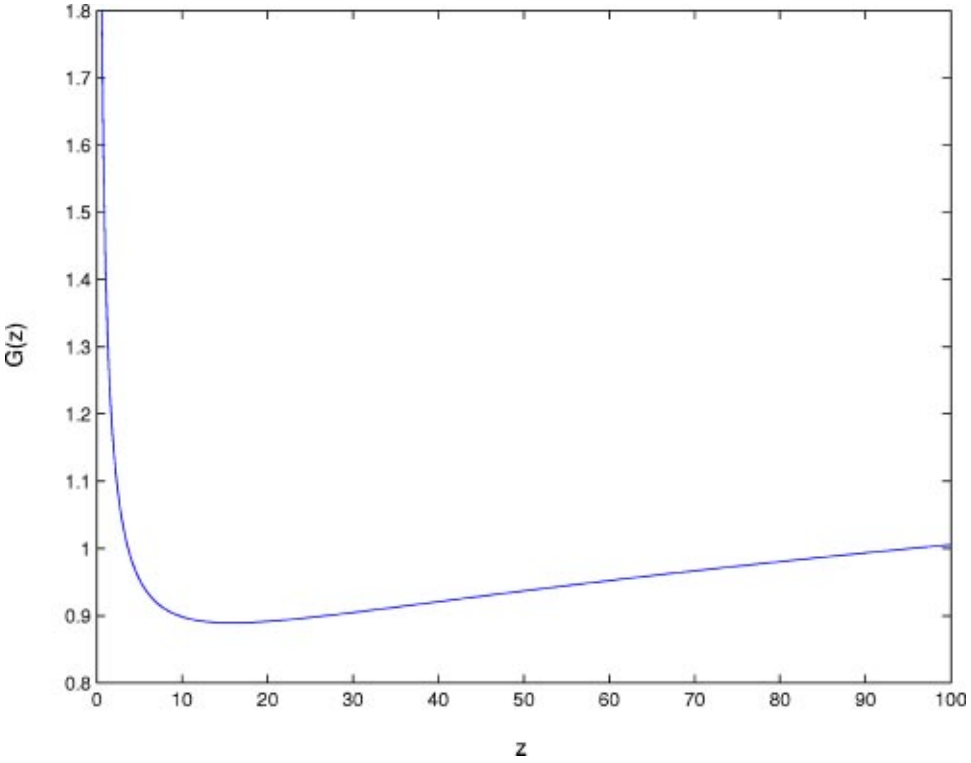

Fig. 1. The gain function $\mathcal{G}(z)$ relating the radiation intensity, frequency, and temperature.

$$
\begin{aligned}
cu &= \int_{\Delta\Omega} d\Omega_{\mathbf{k}} \int_{\Delta\nu} d\nu\, I_\nu \\
&= \int_{\Delta\Omega} d\Omega_{\mathbf{k}} \int_{\Delta\nu} d\nu (B_\nu + D_\nu) \\
&= \int_{\Delta\Omega} d\Omega_{\mathbf{k}} \int_{\Delta\nu} d\nu \frac{h\nu^3/c^2}{e^{h\nu/k_BT}-1} + \int_{\Delta\Omega} d\Omega_{\mathbf{k}} \int_{\Delta\nu} d\nu \;\; D_\nu .
\end{aligned}
\tag{20}
$$

If we change to the variable $x = h\nu/k_BT$, the first integral of the right-hand side of Eq. (20) can be recast as:

$$
\frac{h}{c^2}\left(\frac{k_BT}{h}\right)^4 \int_{\Delta\Omega} d\Omega_{\mathbf{k}} \int_{\Delta x} dx \frac{x^3}{e^x - 1}. \tag{21}
$$

It is clear that letting $\Delta\nu$ go to zero implies that Δx goes to zero also. From Eq. (13) T cannot reach zero except for zero intensity, $I_\nu \to 0$. This limit cannot be reached because we require that this beam have fixed, finite energy density u. Because the integrand is finite, the double integral in Eq. (21) vanishes in the limit of $\Delta\nu$ and $\Delta\Omega$ going to zero.

It follows that the second double integral on the right-hand side of Eq. (20) vanishes in this limit, as does the double integral in Eq. (21). If we use Eq. (21) to rewrite Eq. (20), we obtain

$$
\begin{aligned}
\lim_{\substack{\Delta\nu\to 0 \\ \Delta\Omega\to 0}} \frac{h}{c^2}\left(\frac{k_BT}{h}\right)^4 = \lim_{\substack{\Delta\nu\to 0 \\ \Delta\Omega\to 0}} \Bigg\{ \left(cu - \int_{\Delta\Omega} d\Omega_{\mathbf{k}} \int_{\Delta\nu} d\nu\, D_\nu \right) \\
\times \left[\int_{\Delta\Omega} d\Omega_{\mathbf{k}} \int_{\Delta x} dx \frac{x^3}{e^x - 1} \right]^{-1} \Bigg\} = \infty .
\end{aligned}
\tag{22}
$$

Thus T becomes infinite when a fixed energy is concentrated into a single mode. This limit recovers the special ensemble of a pure state discussed in Sec. IV C, fixed energy with zero entropy. The ensemble degenerates into a single Hilbert space, with an exact photon number $N_{\mathbf{k}} = E/h\nu$ or field mode modulus $a_{\mathbf{k}} = \sqrt{E/h\nu}$. In this limit, these ensemble averages have no statistical uncertainty. The mode fills all of real space and has one wavelength and direction, a fixed absolute amplitude and energy density, but random phase. The phase is independent of energy and does not affect T. This result formally justifies the idealized picture of laser radiation as having infinite temperature.

When radiation is in equilibrium with matter, all Fourier subspace temperatures become the same finite value, returning I_ν to a full blackbody distribution, as in Eq. (18), but valid over all modes. If the matter is selective in its frequency or directional response to radiation, the relaxation of the radiation to equilibrium is similarly limited. (The radiation intensity distribution can be found in general by solving the equation of transfer that describes radiation transport through matter.[10])

D. Temperature versus intensity: Classical limits

For a fixed and frequency-independent $I_\nu \equiv I$, very different temperatures are found at different frequencies. We rewrite Eqs. (12) and (13) in terms of the reciprocal of the mean occupation number, $z \equiv h\nu^3/(c^2 I_\nu) = 1/n$,

$$
k_BT_\nu = h\left(\frac{c^2 I}{h}\right)^{1/3} \mathcal{G}(z), \tag{23}
$$

where

$$
\mathcal{G}(z) = z^{1/3}/\ln(1+z). \tag{24}
$$

The gain function $\mathcal{G}(z)$ is plotted in Fig. 1 and determines how the intensity and frequency are related to the temperature for radiation. $\mathcal{G}(z)$ is singular at $z=0$, with a minimum at approximately $z=15.8$; it grows gradually, unbounded, with increasing z.

Clearly ''hotter'' radiation sources for a given energy are either at low or very high frequencies. For example, a laser in the 600-nm range is less ''hot,'' watt for watt, than an x-ray laser with wavelengths of the order of angstroms. This result makes sense because Eq. (13) must give a constant value for a blackbody, having a minimum in the middle of the frequency domain. Alternatively, the curve represents points where the blackbody distribution crosses a constant specific intensity I. It crosses at two frequencies except when the temperature falls so low that it does not cross at all.

The low- and high-frequency limits of Eq. (23) can be understood differently. If we count photons as particles in terms of their energies $\epsilon=h\nu$, the relation $n=[\exp(\epsilon/k_BT)-1]^{-1}$ is independent of h. In the limit $\epsilon\to\infty$ or $T\to 0$, $n\to\exp(-\epsilon/k_BT)$, the Maxwell–Boltzmann distribution. Photons in this limit act as classical particles, and n tends to be small ($z\to\infty$). Then consider the low-frequency or high-intensity limit of Eq. (23). In terms of intensity, the resulting classical relation $k_BT_\nu=(c^2/\nu^2)I$ is independent of h and arises from a set of classical thermal oscillators. In this Rayleigh–Jeans limit, $z\to 0$ and $n\to\infty$.

Photons therefore have two different classical limits, at high frequencies as particles and at low frequencies as a classical field. The field limit is possible because photons are bosons, and large numbers of photons can coexist in the same field mode. Coherent states can be constructed as analogues to classical field states.

VI. THE TEMPERATURE OF A REAL LASER

Although lasers do not have infinite temperatures, even common, low-power lasers have temperatures closer to those of stellar interiors than to everyday matter temperatures, far exceeding the mistaken equipartition estimates of Sec. II. Lasers operate far into the high-intensity Rayleigh–Jeans limit of Eqs. (12) and (13), where $k_BT_\nu=(c^2/\nu^2)I_\nu$. To see these very high temperatures, the specific intensity needs to be extracted from the laser power P.

The flux density F is related to the specific intensity $I_\nu(\hat{\mathbf{k}})$ by

$$F=dP/dA=\int d\nu\, d\Omega_{\mathbf{k}}(\hat{\mathbf{r}}\cdot\hat{\mathbf{k}})I_\nu(\hat{\mathbf{k}}). \tag{25}$$

The surface area element $dA=r^2d\Omega_{\mathbf{r}}$, and $\cos\theta=\hat{\mathbf{r}}\cdot\hat{\mathbf{k}}$ is the cosine of the angle between the wave vector and surface area normal vector $\hat{\mathbf{r}}$. $d\Omega_{\mathbf{k}}$ is the mode solid angle element, and dA represents the differential exit aperture area of the laser beam.

If $I_\nu(\hat{\mathbf{k}})$ were constant with beam angle and frequency, then $I_\nu(\hat{\mathbf{k}})=I$, and the power from each point of the aperture would be constant over the aperture, $F=P/A$. The power would be

$$P=\pi AI\sin^2\theta_{1/2}, \tag{26}$$

where $\theta_{1/2}$ is the beam spread half-angle, and A is the aperture area of the laser.

Let the specific intensity have the factorized form:

$$I_\nu(\hat{\mathbf{k}})=F_0\Phi(\nu)\mathcal{D}(\hat{\mathbf{k}}). \tag{27}$$

Each function, $\Phi(\nu)$ and $\mathcal{D}(\hat{\mathbf{k}})$, is separately normalized to unity:

$$\int_0^\infty d\nu\,\Phi(\nu)=\int d\Omega_{\mathbf{k}}(\hat{\mathbf{r}}\cdot\hat{\mathbf{k}})\mathcal{D}(\hat{\mathbf{k}})=1. \tag{28}$$

The solid angle variables are θ and φ. They are the polar angle relative to, and the azimuthal angle about, the beam axis, respectively.

We will consider typical line shapes and angular distributions.[1,2] The simplest line shape is the Gaussian form arising from Doppler broadening by thermal agitation of the lasing gas,

$$\Phi(\nu)=\exp[-(\nu-\nu_0)^2/(\Delta\nu_D)^2]/\sqrt{\pi}\Delta\nu_D, \tag{29}$$

where $\Delta\nu_D$ is the half-width due to Doppler broadening:

$$\Delta\nu_D=\nu_0\sqrt{\frac{2k_BT_{\text{gas}}}{mc^2}}, \tag{30}$$

and m and T_{gas} are the mass and temperature of the gas atoms emitting the radiation.

The simplest angular distribution is also a Gaussian from the laser cavity resonating in its fundamental mode:

$$\mathcal{D}(\theta)=2\exp[-2\theta^2/\theta_0^2]/\pi\theta_0^2, \tag{31a}$$

$$\int_0^{2\pi}d\varphi\int_{-1}^{+1}d(\cos\theta)\cos\theta\mathcal{D}(\theta)=1. \tag{31b}$$

The distribution is azimuthally symmetric about the forward beam direction, with $\theta_0\ll 1$ as the half-angle, e^{-2}-power, beam divergence. An ideal laser's beam divergence is diffraction-limited at the aperture: $\theta_0=2\lambda_0/\pi D$, where D equal to the aperture diameter.

Our results are for a red He–Ne gas laser, wavelength $\lambda_0=632.8$ nm or $\nu_0=4.741\times10^{14}$ Hz. We assume a power of $P_0=1$ mW. Figure 2 is a plot of the radiation temperature $T(\nu,\theta)$ for $D=1$ mm and $\Delta\nu_D=0.9$ GHz, corresponding to $T_{\text{gas}}\simeq 390$ K. The beam aperture area A is $\pi D^2/4\simeq 8\times10^{-7}$ m^2, and the total flux density $F_0=P_0/A\simeq 1.3\times10^3$ W/m^2. The beam divergence $\theta_0\simeq 0.4$ mrad.

Figure 3 shows the radiation temperature $T(\nu=\nu_0,\theta)$ for several beam diameters D and corresponding beam divergences θ_0, with the same P_0 and $\Delta\nu_D$. The peak temperature is

$$k_BT_\nu(\nu=\nu_0,\theta=0)=\frac{2c^2F_0}{\pi\sqrt{\pi}\nu_0^2\theta_0^2\Delta_D}. \tag{32}$$

It is independent of D, because the normalization of temperature is independent of D:

$$T\sim F_0/\theta_0^2\sim P_0\theta_0^{-2}/A\sim\theta_0^{-2}D^{-2}\sim D^2D^{-2}. \tag{33}$$

Only the shape of the angular distribution depends on D. Note that T_ν diverges if Δ_D or $\theta_0\to 0$, reproducing the pure state of Sec. V C.

To infer the radiation temperature of a laser requires knowing the specific intensity I_ν of its light. Because a laser's radiation is so intense, we can use the Rayleigh–Jeans

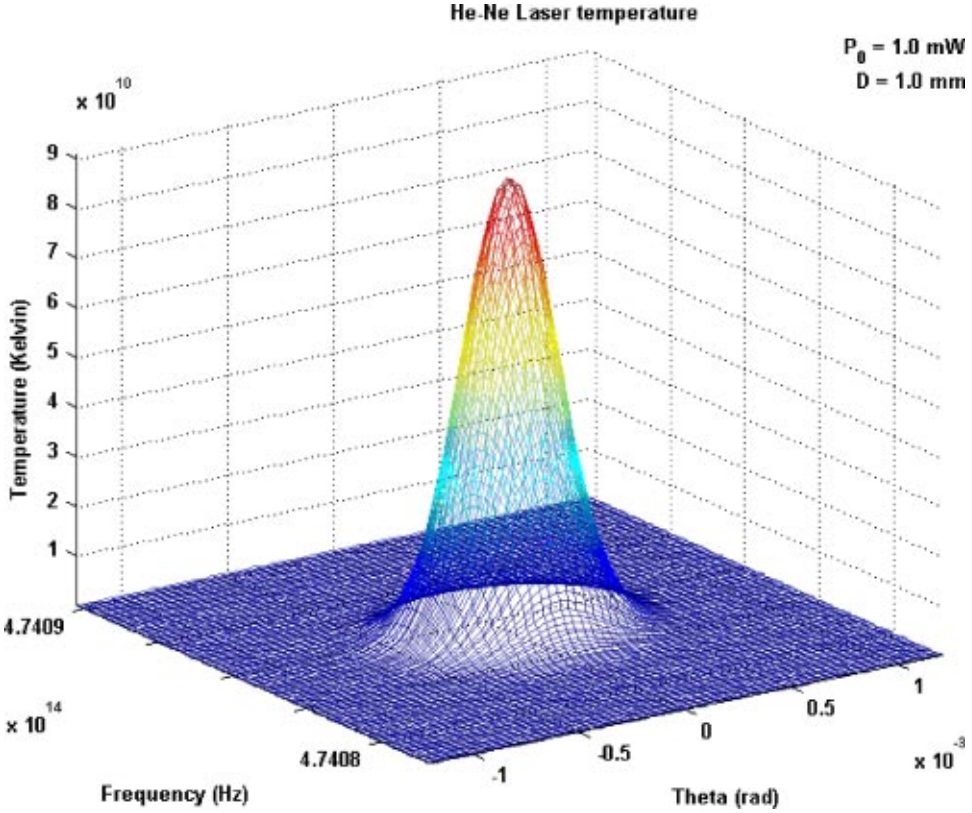

Fig. 2. The helium–neon laser radiation temperature $T(\nu,\theta)$ for $D=1\,\mathrm{mm}$, $\Delta\nu_D=0.9\,\mathrm{GHz}$, and $P_0=1.0\,\mathrm{mW}$.

limit of Eq. (13): $k_B T_\nu=(c^2/\nu^2)I_\nu$. Finding I_ν requires making separate geometry- and frequency-based measurements. The former requires an intensity measurement at a fixed frequency, typically with a photodetector. The latter determines the spectral characteristics of the beam.[2,16,17]

We place a photodetector of known area A and frequency response or measurement efficiency $R(\nu)$ orthogonally across the beam and centered along the beam axis ($\theta=0$), at a distance r from the beam exit. $R(\nu)$ is normalized so that its integral is unity. The photodetector measures an incoming power P:

$$P=2\pi A\int_0^\infty d\nu\int_A d(\cos\theta)\cos\theta R(\nu)I_\nu(\nu,\theta), \tag{34}$$

using the definition (25). The factor of 2π represents the azimuthal integration around the beam axis and assumes axial symmetry of the beam's intensity.

We make the following simplifying assumptions.

(1) The frequency and angular profiles are Gaussian, like Eqs. (29) and (31). From these profiles, we can infer the peak flux density F_0 with a few measurements.
(2) The solid angle subtended by the photodetector is $\Omega=A/4\pi r^2$. The photodetector itself is circular, and the half-angle θ_A it subtends with the beam axis is given by $\tan\theta_A=\sqrt{A/\pi r^2}$.
(3) All relevant angles, θ_A and θ_0, are small. Then $\cos\theta\simeq 1-\theta^2/2$. The angular integration becomes:

$$\int_{\cos\theta_A}^1 d(\cos\theta)\cos\theta I_\nu(\nu,\theta)\doteq\int_0^{\theta_A}d\theta\,\theta I_\nu(\nu,\theta). \tag{35}$$

(4) The frequency integration can be simplified if we assume that $R(\nu)$ is constant across a range ΔF, $R(\nu)=(\Delta F)^{-1}$. If we do the frequency integration numerically, however, this simplification is not necessary.

We can use the values of ν_0, $\Delta\nu_D$, and θ_0 that are specified by the laser's manufacturer. With additional instruments, we can measure these quantities, although such measurements are more difficult than the power measurement. For example, we can determine the frequency profile and infer ν_0 and $\Delta\nu_D$ by measuring the specific intensity as a function of wavelength. The light wavelength is typically measured with a grating spectrometer. But a grating does not have the frequency resolution needed for a laser, and instead we should use a Fabry–Perot interferometer-based analyzer. We can determine the angular profile and infer θ_0 by measuring how the relative intensity varies as we vary r, the distance of the photodetector from the beam aperture. Because we move the photodetector further away from the laser, it subtends a larger half-angle θ_A with the beam axis.

If we substitute these derived values and assumed simplifications into Eq. (34), we can do the angular integral analytically. The frequency integral can be done numerically. With the measured values of P, A, ν_0, $\Delta\nu_D$, and θ_0, we can infer F_0 and $I_\nu(\nu,\theta)$ from Eqs. (27), (29), (31), and (34). Our results are in Figs. 2 and 3; your results should be similar unless your laser is not operating in its fundamental frequency mode. (If the photodetector is calibrated in absolute power units, the value of F_0 would also be in absolute units. If it is not possible to measure P in absolute units, we can

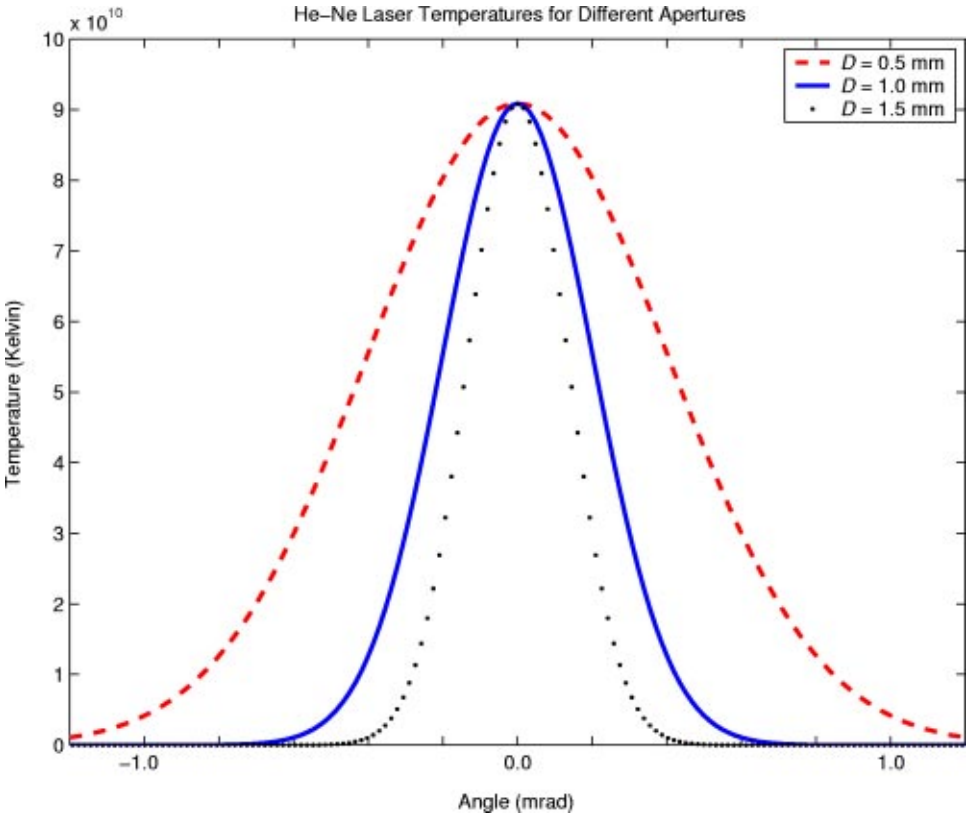

Fig. 3. The helium–neon laser radiation temperature $T(\nu_0,\theta)$ for three aperture diameters D. The intensity/temperature profile narrows as D increases and the beam becomes more parallel.

still measure a relative intensity profile in frequency and angle, normalized to the peak intensity. The resulting temperature also would be in relative units.)

VII. OTHER PHYSICAL CONSEQUENCES

Nonequilibrium radiation temperature and its large magnitude in lasers have important theoretical and practical consequences. Consider the entropy production density of radiative transfer (matter–radiation coupling). The matter interacting with the radiation has a local temperature T. This entropy production density can be related to the radiation energy specific intensity I_ν and an entropy specific intensity J_ν:[4]

$$
\begin{aligned}
&-\frac{\dot{u}+\boldsymbol{\nabla}\cdot\mathbf{F}}{T}+\dot{s}_r+\boldsymbol{\nabla}\cdot\mathbf{H}\\
&\quad=\int d\nu\, d\Omega_{\mathbf{k}}\left\{-\left(\hat{\mathbf{k}}\cdot\boldsymbol{\nabla}I_\nu+\frac{1}{c}\dot{I}_\nu\right)\frac{1}{T}+\left(\hat{\mathbf{k}}\cdot\boldsymbol{\nabla}J_\nu+\frac{1}{c}\dot{J}_\nu\right)\right\}\\
&\quad=\int d\nu\, d\Omega_{\mathbf{k}}\left\{\left(\hat{\mathbf{k}}\cdot\boldsymbol{\nabla}I_\nu+\frac{1}{c}\dot{I}_\nu\right)\left(\frac{\partial J_\nu}{\partial I_\nu}-\frac{1}{T}\right)\right\}\\
&\quad=\int d\nu\, d\Omega_{\mathbf{k}}\left\{\left(\hat{\mathbf{k}}\cdot\boldsymbol{\nabla}I_\nu+\frac{1}{c}\dot{I}_\nu\right)\left(\frac{1}{T_\nu(\hat{\mathbf{k}})}-\frac{1}{T}\right)\right\}. \qquad (36)
\end{aligned}
$$

H and **F** are vector densities of radiative entropy and energy flux, respectively, and u and s_r are radiative energy and entropy densities, respectively. The radiation temperature $T_\nu(\hat{\mathbf{k}})$ is defined in Eqs. (7), (13), and (19). The first term on the first line is the matter entropy production density; the second and third are the radiation entropy production density.

Equation (36) is positive semidefinite and expresses the second law of thermodynamics in radiative transfer. The left-hand factor in the last integrand of Eq. (36) is the energy transfer rate density, which is equal and opposite for matter and radiation.[10] For strictly thermal emission, this factor corresponds to the net cooling or heating of matter due to the radiation beam in the direction $\hat{\mathbf{k}}$.

- If $T>T_\nu(\hat{\mathbf{k}})$, it is positive: the matter cools, the field gains energy, and the entropy production is positive.
- If the temperature inequality is reversed, the sign of this factor also reverses: the field loses energy, the matter is heated, and the entropy production remains positive.
- The product in the integral is zero only when the matter and radiation temperatures are equal and no transfer occurs.

The two other ways that matter can transfer energy to other matter are by convection and diffusion. These mechanisms require matter–matter contact and gradients in intensive variables such as temperature. Radiative energy transport, on the other hand, depends only on the difference of the local matter and radiation temperatures at a single point in space. Even a low-power laser is very effective at transferring energy, because its radiation temperature is so large. Energy transfer dominated by radiative processes is minimally affected by convection and conduction if the matter temperature is much smaller than the radiation temperature.

Practical consequences of these results include the following. Because of the high beam temperature, radiative transfer dominates over the other, more destructive and undesirable heat transfer processes of convection and conduction, making lasers effective tools for surgery. There is neither the time nor the energy to induce gradients large enough to make

convection or conduction important. The overall deposited energy is small, and the cut tissue suffers little damage. The cooling of atoms with lasers is also interesting in light of the results of this paper.

In Sec. IV C, two cases of vanishing entropy were considered. Case 2 represents the laser radiation cooling the atoms, while Case 1 conforms to the behavior of the atoms being cooled. When the atoms decrease in entropy, according to Case 1, their temperature and energy must also go down. But the radiation behaves according to Case 2: its temperature also goes down, but its entropy increases instead, while its power stays constant.

The classical field limit of $T_\nu(\hat{\mathbf{k}})$ is obtained by letting $h \to 0$ in Eq. (13) and holding everything else constant. The result, $k_B T_\nu(\hat{\mathbf{k}}) = h\nu n = c^2 I_\nu(\hat{\mathbf{k}})/\nu^2$, is identical to the high-intensity or low-frequency limit, independent of h if written in terms of $I_\nu(\hat{\mathbf{k}})$. The magnitude of the flux density $|\mathbf{F}| = F$ is

$$F = \int d\nu\, d\Omega_{\mathbf{k}} k^2 (\hat{\mathbf{r}}\cdot\hat{\mathbf{k}}) k_B T_\nu(\hat{\mathbf{k}}). \tag{37}$$

This result can be compared to the antenna or noise temperature familiar in radio-frequency electromagnetism.[18] In a one-dimensional system, the power P_{noise} of a pure noise signal is associated with a temperature T_{noise}:

$$P_{\text{noise}} = \int_0^\infty d\nu\, k_B T_{\text{noise}}(\nu). \tag{38}$$

The result (37) is the three-dimensional, solid-geometry analogue of the radio-frequency antenna or circuit result (38). It should be stressed that the antenna temperature is the temperature of the radiation, not the antenna itself; just as $T_\nu(\hat{\mathbf{k}})$ is not the temperature of the lasing gas, but of the laser light.

The arguments of Sec. V make clear that the noise temperature defined in Eqs. (37) and (38) requires a randomized phase distribution for the noise signal, in keeping with the usual intuitive definition of noise.

The results of this paper demonstrate some of the physical implications of temperature for nonequilibrium systems, in particular for radiation and laser light. More consequences can be inferred by applying the techniques and results presented here to other aspects of radiation, radiation transport, and lasers, and can be explored starting with the references.

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[12]The most general coherent state representation of $\hat{\rho}$ is $\int d^2\alpha d^2\beta R(\alpha,\beta)|\alpha\rangle\langle\beta|$, but this form is not necessary for our purposes (Ref. 1).
[13]This result reflects the complementarity of photon number and field phase. The quantum phase (Susskind–Glogower) operator $\exp(i\hat{\phi}) \equiv \hat{a}/\sqrt{\hat{a}\hat{a}^\dagger} = \Sigma_n |n\rangle\langle n+1|$ is nondiagonal in the $|n\rangle$ basis. The commutator $[\exp(i\hat{\phi}), \hat{n}] = \exp(i\hat{\phi}) \neq 0$. See Ref. 14 for a detailed exploration of radiation phase in the quantum case.
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