EPJ manuscript No.
(will be inserted by the editor)

What Conditions Make Minimum Entropy Production Equivalent to Maximum Power Production?

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Abstract. Optimization of processes can yield a variety of answers, depending not only on the objective of the optimization but also on the constraints that define the problem. Within the context of thermodynamic optimization, the role of the constraints is particularly important because, among other things, their choice can make some objectives either equivalent or inequivalent, and can limit or broaden the possible kinds of processes one might choose. After a general discussion of the principles, a specific example of a model power plant is analyzed to see how the constraints govern the possible solutions.

1 Introduction

In the context of optimizing processes by thermodynamic methods, one may choose from a variety of objective functions, particularly if the process transpires in finite time and is irreversible. Recent literature has addressed some aspects of the equivalence of some of these functions, sometimes in a vociferous, even heated manner [1–4]. In particular, there is an apparent controversy over the equivalence or inequivalence of minimizing entropy production and maximizing power production.

The immediate purpose of this note is to resolve this specific issue by showing how the choice of constraints determines the answer. Minimum entropy production and maximum power are inequivalent in general but are equivalent under certain specific conditions. The larger goal is to illustrate the importance of specifying constraints explicitly and of demonstrating their importance in the teaching of thermodynamics. Recognizing the importance of constraints in thermodynamics is hardly a new issue, but the regularity with which apparent paradoxes and controversies occur because of misunderstood constraints indicates that illustrating their roles in new situations will always have pedagogic value.

An intuitive example to illustrate the difference is driving a car under two very different conditions to travel the same distance: in one case, if you are likely to be late for an appointment 5 miles away, you will maximize power and sacrifice the entropy and energy of the fuel; in the other, if you are low on gas but must travel 5 miles to the nearest gas station, you will minimize fuel consumption and, very likely, entropy production, but will certainly not maximize power! If there are other constraints, such as a speed limit or an approaching closing time at the gas station, you will presumably choose some compromise between maximizing power and minimizing entropy production.

Suppose we view the operation of a heat engine as an economic process that converts fuel at its market price into energy at its market price. Presumably, we would like to optimize the net revenue; then there is a region of operating conditions for our engine, ranging from maximizing power to minimizing entropy production, within which all the possible optimum conditions lie. Where we choose to operate depends on the relative prices of fuel and power [5,6]. At one extreme we operate the engine to deliver as much power as possible without regard to how much fuel we waste. At the other extreme we try to get the maximum work out of the fuel without regard to how long it takes, i.e. to use the fuel as efficiently as possible.

Without further constraints, both of these objectives often have uninteresting answers for many model systems. The maximum power objective often calls for fuel consumption at an infinite rate, while the minimum fuel consumption (and minimum rate of entropy production) calls for infinitely slow operation. This is the case for one model [1] which we discuss below. For many other models with adequate constraints, these objectives give rise to well defined, finite modes of operation. The surest way to make the minimum entropy production rate problem well defined is to require that something happen; otherwise having nothing happen is inevitably the best solution.

In the real world, we encounter not only a variety of objectives to optimize; we must also deal with a variety of constraints, depending on the particular problem we are addressing. For the example of driving a car just discussed, the most natural constraint is the imposition of a fixed length of the trip or the traversal of a given route; in that case the two problems just presented become doing this trip either in minimum time or with minimum fuel consumption. For a real automobile these problems give distinct and finite solutions. Another way to constrain this problem, the choice of the vacationers, is to impose an inequality, rather than an equality: the trip cannot be shorter than the shortest route, but if there are factors that make side trips valuable, the shortest route may not be the optimum.

Still another way to constrain the same problem is to restrict the analysis to a fixed amount of fuel consumed [2, 1, 7]. This does indeed make the problem well defined in the above sense, and can also, with a suitable additional assumption, make equivalent the particular two optima of maximum power and minimum rate of total entropy production. Constraining the amount of fuel consumed however is not the only "consistent" or useful way to make the problem well-defined, as has been shown in a variety of situations [6]-[30].

Whether the additional assumption mentioned just above is "suitable" depends on whether the constraints specifying the process completely determine the final states of all the participating systems apart from the tradeoff of energy between a work reservoir and a heat reservoir. If they do, then minimum entropy production and maximum power are indeed equivalent. This follows immediately from the equality form of the Second Law [31–33]

$$T_L \Delta S_{universe} \equiv -\Delta A_{universe} \tag{1}$$

where T_L is the temperature of the atmosphere (environment), S is the total entropy, and A is the availability with respect to the atmosphere. We separate $\Delta A_{universe}$ into a portion going to the work reservoir, a portion going to the atmosphere, and a term ΔA_{spent} which includes the availability changes of all other systems. More specifically,

$$\begin{aligned}\Delta A_{universe} &= \sum_{i\in\{allsystems\}} \Delta A_i \\ &= \sum_{\substack{i\in\{allsystems\},\\ i\neq WorkRes,\\ i\neq atmosphere}} \Delta A_i + \\ &\quad + \Delta A_{WorkRes} + \Delta A_{atmosphere} \\ &= \Delta A_{spent} + W + 0 \end{aligned}\tag{2}$$

where ΔA_{spent} is the sum of ΔA_i over all systems other than our work reservoir and the atmosphere, W equals $\Delta A_{WorkRes}$, and

$$\begin{aligned}\Delta A_{atmosphere} &= \Delta U_{atmosphere} - T_L \Delta S_{atmosphere} \\ &= 0\end{aligned}\tag{3}$$

since availability is counted with reference to the atmosphere. Using our definition of ΔA_{spent}, we can reexpress equation (1) as

$$-\Delta A_{spent} = T_L \Delta S_{universe} + W. \tag{4}$$

This expression for ΔA_{spent} makes it clear that once ΔA_{spent} is fixed, maximizing one term on the right hand side of equation (4) and minimizing the other achieve the same thing. Specifying the initial and final states of all systems other than the work reservoir and the atmosphere, but including any reservoirs that supply availability, specifies ΔA_{spent} and thus maximizing the work W (or work per unit time $\dot{W}$, or average work per unit time) is equivalent to the minimizing the total entropy change $\Delta S_{universe}$ (or the rate of entropy change $\dot{S}_{universe}$, or the average rate of entropy change, depending on which variable one wishes to use). The equivalence found between these two objectives in references [1–3] comes about by fixing $\dot{A}_{spent}$ whose value is determined by the assumption that fuel is consumed at a given rate and the combustion products cool to the temperature of the atmosphere. It is, however, neither necessary nor, in many situations, relevant or desirable to specify the final state of the surroundings – fuel not burned today can usually be saved for tomorrow.

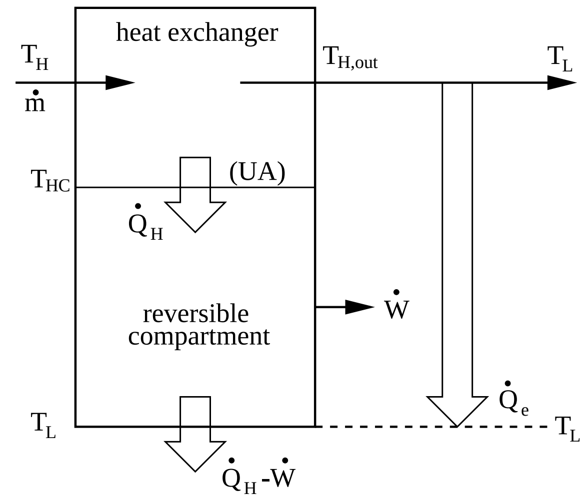

Fig. 1. Model power plant described in section 2.

2 A Model Power Plant

To take the discussion to a more concrete level, we analyze a model of a power plant [1] shown in figure 1. The irreversible heat engine for our model power plant works between a hot stream having temperature T_H and mass flow rate $\dot{m}$, and a heat reservoir (the environment) at temperature T_L [34]. The model consists of a power-producing compartment and compartments that provide heat input and removal to drive a power-producing cycle. The power-producing compartment operates reversibly and is connected to the hot stream through heat conductance UA, in the notation of Ref. [1]. In the course of the heat exchange, the hot stream is cooled to $T_{H,out}$ while transferring the heat flux

$$\dot{Q}_H = \dot{m} c_p (T_H - T_{H,out}) \tag{5}$$

to the power producing compartment at temperature T_{HC}. c_p is the specific heat capacity of the hot stream. Following Bejan [1], we also take this heat flow to be equal to

$$\dot{Q}_H = UA(T_{H,out} - T_{HC}) \tag{6}$$

which reflects the finite heat conductance between the hot stream and the power producing compartment [35]. Eliminating $T_{H,out}$ from these two equations leads to a heat conduction law relating the temperature of the working fluid and the initial temperature of the hot stream,

$$\dot{Q}_H = \frac{\dot{m}c_p UA}{\dot{m}c_p + UA}(T_H - T_{HC}). \quad (7)$$

The power plant produces work at the rate $\dot{W}$ and therefore rejects heat

$$\dot{Q}_L = \dot{Q}_H - \dot{W} \quad (8)$$

to the atmosphere at temperature T_L, and consequently with availability of zero. We assume reversible operation inside the engine working between T_{HC} and T_L giving

$$\dot{W} = \dot{Q}_H(1 - \frac{T_L}{T_{HC}}). \quad (9)$$

Expressing this entirely in terms of the temperatures with the aid of equation (7) gives

$$\dot{W} = \frac{\dot{m}c_p UA}{\dot{m}c_p + UA} T_L (1 + \frac{T_H}{T_L} - \frac{T_H}{T_{HC}} - \frac{T_{HC}}{T_L}). \quad (10)$$

Next we evaluate the entropy production in the process by summing the entropy changes of the participating systems. Since the engine undergoes steady operation, it has no net entropy change. Thus we need to evaluate only the entropy changes of the hot stream and of the atmosphere. Here again there seems to have been some confusion in the literature concerning ways to do this evaluation, which we take up more fully in the next section. For the present, following Bejan [1], we assume that the hot stream which exits the heat exchanger compartment at temperature $T_{H,out}$ is rejected into the atmosphere at temperature T_L, and thus any residual availability remaining in the stream is completely degraded. This involves an additional heat flow $\dot{Q}_e = \dot{m}c_p(T_{H,out} - T_L)$ from the hot stream to the atmosphere at temperature T_L. The total entropy production then becomes

$$\dot{S}_{universe} = \frac{\dot{Q}_L}{T_L} + \frac{\dot{Q}_e}{T_L} + \\ + \dot{m}(s_{stream}(T_L) - s_{stream}(T_H)), \quad (11)$$

where $s_{stream}(T) = s_0 + c_p \ln(T)$ is the specific entropy of the hot stream as a function of the temperature. Substituting from equation (8), we have our desired expression for the total rate of entropy production,

$$\begin{aligned} \dot{S}_{universe} &= -\frac{\dot{W}}{T_L} + \frac{\dot{Q}_H + \dot{Q}_e}{T_L} + \dot{m}\Delta s_{stream} \\ &= -\frac{\dot{W}}{T_L} - \dot{m}\frac{\Delta u_{stream} - T_L \Delta s_{stream}}{T_L} \\ &= -\frac{\dot{W} + \dot{m}\Delta a_{stream}}{T_L} \end{aligned} \quad (12)$$

where Δu_{stream} and Δa_{stream} are the specific internal energy and availability changes of the stream cooling from T_H to T_L. Note that this is essentially equation (1) again for this specific example.

Equation (12) shows specifically that once the burn rate $\dot{m}$ is specified, we are left with a situation of the type described at the end of Section 1, with equivalence between minimum entropy production rate and maximum power. At fixed $\dot{m}$, we have only the degree of freedom T_{HC} in $\dot{W}$ in equation (10). This gives the familiar result

$$T_{HC} = \sqrt{T_H T_L}, \quad (13)$$

with the efficiency $\eta = \dot{W}/\dot{Q}_H$ at maximum $\dot{W}$ given by the Curzon-Ahlborn expression $1 - \sqrt{T_L/T_H}$.

Once we reduce the problem to this one degree of freedom, the important ability to vary the rate of fuel burning, maybe at lower efficiency, is missing. If we allow the burn rate $\dot{m}$ to vary, the situation is very different. In that case maximum $\dot{W}$ is achieved for $\dot{m} = \infty$ while minimum $\dot{S}_{universe}$ is achieved for $\dot{m} = 0$. If the problem is better specified, for example by requiring that $\dot{m}_{min} \le \dot{m} \le \dot{m}_{max}$, maximum power occurs for

$$\dot{m} = \dot{m}_{max}; \qquad T_{HC} = \sqrt{T_H T_L}, \quad (14)$$

while minimum entropy production occurs for

$$\dot{m} = \dot{m}_{min}; \qquad T_{HC} = \sqrt{T_H T_L}. \quad (15)$$

These are typical, if trivially simple, examples of *boundary extrema*, in which the optimum conditions correspond to operating at a limit of a constraint.

3 Ways to Evaluate

One central feature of this example involved the way that we evaluated the availability of the hot stream as it exits from the heat exchanger compartment. The analysis above assumed that all of this availability was discarded. If this is not the case, the conditions of minimum entropy production rate and maximum power can be different, even for fixed $\dot{m}$. There are many examples for which it is inappropriate to include the discarded availability term $\dot{Q}_e/T_L$. This is the case, for example, in solar power plants in which the hot stream can be "recycled", i.e. heated again by the sun rather than discarded into the atmosphere. The term is also inappropriate for a device operating from a reservoir. In this case a simple heat conduction to the power producing compartment does not require the disposal of Q_e to the atmosphere. The point of view in references [1–4] is that there is only one "freely variable" reservoir (the atmosphere), i.e. any other reservoir needs to be restored by burning fossil fuels. This in turn brings us back to the earlier analysis including Q_e/T_L. Restoring the other reservoir is by no means a necessity. Geothermal power plants are industrial examples of this mode of operation. A dramatic example suitable for demonstration purposes is provided by James Senft's palm-size Stirling engines [36] which can run for about half an hour using a small glass of ice water as the other reservoir.

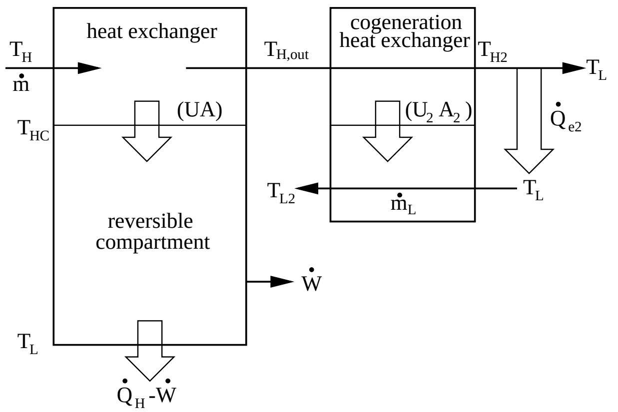

Fig. 2. Model of a cogeneration power plant. The exhaust stream at temperature $T_{H,out}$ heats a cold stream at T_L.

If we do not include the discarded availability term in equation (12), the entropy production rate becomes, instead,

$$\begin{aligned}\dot{S}_{universe} &= \dot{Q}_L/T_L + \dot{m}(s_{stream}(T_{H,out}) - s_{stream}(T_H)) \\ &= \dot{Q}_H/T_{HC} + \dot{m}c_p \ln(T_{H,out}/T_H). \qquad (16)\end{aligned}$$

The value of T_{HC} which minimizes this for constant $\dot{m}$ is certainly different from the one given in equation (13); it gives an optimal $T_{HC} = T_H$ which results in zero values for $\dot{Q}_H$ and $\dot{W}$ and so corresponds to the "nothing happen" solution mentioned in the first Section. To get a more meaningful result, we must require that something happen. Many studies have done just that for reciprocating engines by requiring that the engine carry out one cycle, sometimes in a fixed period [6]-[9], [12,13,16], [18]-[24], [27]-[30]. In the present example this has no easy counterpart since the working fluid in the engine has been abstracted away. One can expand the model however and require a certain amount of the working fluid to traverse the cycle. Since this would lead us in a direction already well-explored by the references just cited, we do not pursue this direction further here.

Another important example in which the entropy production should be counted differently is cogeneration power plants. In these plants the waste heat is used to heat some cold stream, typically for district heating. Consider a modification of our example shown in figure 2 in which the exhaust stream at temperature $T_{H,out}$ is fed to a second heat exchanger with conductance U_2A_2 which provides contact with a stream coming in at temperature T_L flowing at a rate $\dot{m}_L$ with constant specific heat capacity.

In this model, some of the availability in the hot stream is stored in the cold stream rather than dumped into the atmosphere. While $\dot{W}$ remains unchanged, the entropy production rate now becomes a sum of three terms representing the rates of change of the entropy of the atmosphere, the hot stream, and the cold stream:

$$\begin{aligned}\dot{S}_{universe} &= \frac{\dot{Q}_L + \dot{Q}_{e2}}{T_L} + \dot{m}(s_{stream}(T_L) - s_{stream}(T_H)) \\ &+ \dot{m}_L(s_{coldstream}(T_{L2}) - s_{coldstream}(T_L)) \qquad (17)\end{aligned}$$

where the exhaust heat $\dot{Q}_{e2}$ is now given by

$$\dot{Q}_{e2} = \dot{m}c_p(T_{H2} - T_L) \qquad (18)$$

and T_{L2} and T_{H2} are the temperatures of the two streams exiting from the extra heat exchanger used in the cogeneration. The equations describing this second heat exchanger are straightforward but messy. At fixed $\dot{m}$, optimization with respect to T_{HC} is possible to carry out analytically although the procedure involves solving a cubic equation and yields a rather long expression. The resulting $T_{HC}^{min\dot{S}_{universe}}$ is always between $T_{HC}^{max\dot{W}}$ and T_H, approaching these values only as the constrained value of $\dot{m}$ approaches infinity or zero, respectively. Varying $\dot{m}$ gives the familiar result that for minimum $\dot{S}_{universe}$ we want $\dot{m}$ as small as allowed while for maximum $\dot{W}$ we want $\dot{m}$ as large as allowed.

Which modes of entropy production one counts depends on the purpose of the study. In some problems, minimum entropy production rate and maximum power are equivalent objectives; in others they are not. One sufficient condition for the equivalence of these two objectives is that the final states of all participating systems except one work reservoir and the atmosphere (defining availability) be given and fixed. In this case, equation (1) fixes the sum of $\dot{W} + T_L\dot{S}_{universe}$ and maximizing one term is equivalent to minimizing the other. If the final states of other participating systems are not fixed by our constraints this need not be the case. In fact, it follows easily from equation (1) that either all three of the quantities $\dot{A}_{spent}$, $\dot{W}$, and $\dot{S}_{universe}$ are stationary at a point x^{opt} in control space or at most one is. To see this, consider a point x^{opt} where one of these quantities is stationary and hence its directional derivative in every feasible direction is zero. Our result follows by differentiating both sides of equation (1) and noting that two of the three terms cannot vanish without the third one vanishing. Thus if there is a feasible variation allowed by the constraints on our controls which changes one of the three terms to first order, it must also change at least one other one. In particular, conserving fuel changes $\dot{A}_{spent}$ to first order and, as long as such variations are not ruled out by the constraints of the problem, power and entropy production rate cannot both be stationary for such variations.

4 Conclusions

We have shown that the equivalence of minimum entropy generation and maximum power is limited to rather special constraints and ways of counting the entropy produced. There are certainly situations where these constraints and ways of counting are appropriate but there are many others [37]. One can however give a general rule of thumb. Minimizing the rate of entropy production and maximizing the power both push an operation toward minimum waste along any axis whose scale measures wastefulness. However these same optimizations push in opposite directions along any axis whose scale measures frugality. This rule of thumb is the basis of Avraham Nitzan's characterization of minimum entropy production as the objective of the conservationist and maximum power as the objective of the industrialist. The two versions of the toy power plant model demonstrated how this comes about.

One important reason for studying thermodynamic processes in finite time is to pursue the quest for understanding the limits of what can physically be achieved in such processes quite aside from any direct implications for engineering. The simplest analyses of this kind reveal how different sources of irreversibility individually influence the limits to power generation and entropy production. It is now clear that some systems have well-defined regimes in which one or another source of irreversibility dominates and sets the limits of performance (12); in such systems, the results of a complex analysis including all sources of irreversibility tell us essentially nothing more than separate analyses based on individual sources of irreversibility. We may expect other systems to show richer, nonlinear behavior, in situations in which different sources of irreversibility interact with one another. Whether simple or complex, these limits serve to quantify the intrinsic limitations on energy conversion and transfer in finite time.

We conclude by reminding the reader that thermodynamics is about much more than power plants and engines. Its domain includes limitations to the performance of such diverse processes as lasing, photochemical synthesis and stellar collapse. Many of these limitations are obtained by calculations which model the processes as heat engines [29,27,28]. Thus there is much to be learned from evaluating limits to energy conversion while considering a host of possible loss mechanisms, subject to a host of possible constraints and optimizing a host of possible objectives. The fact that synthesizing these possibilities into a unified scheme still remains a challenge is not a reason to restrict the set of allowed problems. Rather, it is a challenge to continue the development of thermodynamics as the physics of limits to what is possible.

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34. In Bejan's discussion [1] the hot stream is the result of a combustion process. For simplicity, we have omitted this complication.
35. This way of counting the heat corresponds to unmixed combustion [1]. The model of well mixed combustion leads to the same results with the slight difference that the proportionality between $\dot{Q}_H$ and $T_H - T_{HC}$ becomes $\dot{m}c_p(1-exp(-UA/\dot{m}c_p))$ instead of $\dot{m}c_pUA/(\dot{m}c_p+UA)$ obtained in equation (7). Both of these cases lead to the same optimal T_{HC} which is independent of $\dot{m}$. Intermediate mixing models generically lead to more complicated results in which the optimal T_{HC} depends on $\dot{m}$.
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37. See reference [30] for an example in which maximum power is equivalent to *maximum* entropy production rate.