

Thermal Mass and Thermal Inertia in Fluids — A Comparison of Hypotheses

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Abstract. Thermal inertia and thermal mass are concepts that offer means to describe transport of heat in nonequilibrium fluids. However there are options regarding how to separate the part of the total flow that carries entropy from the “mechanical”, nonentropy-bearing part. Some hypotheses are examined and compared for constructing such a field theory of thermal mass in the energy representation. A global intrinsic symmetry and a finite thermal momentum imply that any formulation which hypothesizes a constant ratio θ of thermal mass to the entropy must tie the thermal mass to the so-called bare mass of particles, to preserve the global conservation of matter. However, in any formulation consistent with the Grad-Boltzmann theory, where θ must be variable, the thermal mass behaves as a separate variable governed by the entropy and the second law. Nonetheless, in this case θ has a reasonably broad plateau of values within which entropy is a measure of the thermal mass associated with changes of state. Nonlinear transformations linking usual thermodynamic variables with those of the thermal mass frame preserve the components of the tensor of matter, including Nöther's energy and pressure. A formula is given for the fraction of the observed mass assignable as thermal mass, in accordance with Grad's solution of the Boltzmann equation.

1. Introduction

The thermal inertia, *a gradual change of the heat flux under a rapid change of the temperature gradient*, comes logically from kinetic theory in Grad's moment analysis of the Boltzmann equation [1] and has been rediscovered in various modifications of the Chapman-Enskog [2, 3] and moment methods [4–7]. The effect manifests itself in both energy and entropy representations of thermodynamics, in the latter case as an inertia of the heat, and in the former as a momentum of the entropy flow. While implicit in early kinetic approaches [8–10], the effect became apparent through paradoxes arising from the parabolic equations of heat implying infinite propagation speeds of thermal signals [11–18]. Experiments with heat waves in liquid helium are manifestations of thermal inertia [19–21]. However predictions say that this effect should also manifest itself in dielectric solids

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[22–24]; experiments do confirm the propagating-wave nature of heat in the so-called “ballistic” regime of phonon transfer [25]. Probably its full complexity can be described adequately only as quantum transport [26] which is perhaps why classical relativistic approaches [27–29] did not add significantly to our understanding of the effect. Nevertheless it has been suggested [30] that the effect might be essential for solving the long-standing difficulty of the relativistic temperature transformation [31].

Inclusion of thermal inertia had a remarkable influence on the thermodynamic analyses of heat [32–35], and on the form of the higher-order hydrodynamics [36–40]. In the main, these studies shifted attention from the nonequilibrium internal energy to the kinetic (flux-dependent) entropy, typical of statistical mechanical approaches, leaving the physical problem of thermal inertia unexplained. It is useful to review the situation of the momentum of heat flow, before proceeding further. In both representations, energy and entropy, at least one sort of assignment of an inertia to the heat flux is easy to make, because any flow of the (relativistically evaluated) energy \mathbf{j}_q must correspond to the momentum density \mathbf{j}_q/c^2 , according to relativity theory. While this is true, it is incomplete because the “robust”, persisting momentum \mathbf{j}_q/c^2 is only a *resulting*, residual quantity, fixed by the definition of the hydrodynamic velocity in the particle frame, in contrast to the energy frame where \mathbf{j}_q and its associates would not appear. Within constraints imposed by the covariance principles and symmetry of the matter tensor, the mechanism of heat transfer involves diffusive momenta of different magnitudes, and \mathbf{j}_q/c^2 is only the net result of their incomplete compensation in the relativistic particle frame. Hence the propagation speed, $3^{-1/2}c$ is very high (the related residual momentum, \mathbf{j}_q/c^2 , is small), whereas the actual signal speeds, within Grad’s theory, are low, of order kT/m , the thermal speed. Paradoxically, while the value \mathbf{j}_q/c^2 could only be associated with nearly reversible processes, actual flows clearly generate large entropy, corresponding to the finite values of thermal conductivities of the fluid.

The present work and a complementary relativistic approach develop a quantitative description of thermal inertia, yet leave some ambiguities unresolved. These will remain until some basic problems in nonequilibrium theory are solved. For example, Callen’s postulational formulation of thermodynamics [41] and the correspondence of various representations have no clear extensions to nonequilibrium theories as yet. Inertia of flowing mass and energy are not equivalent in that context. The inertia of mass diffusion follows from nearly classical concepts: the arguments may be kinetic [42, 43], random walk [44], or even phenomenological [45–47]. They can apply to molecular and Brownian diffusion [48, 49], and to turbulent systems [50]. Otherwise, the arguments pertaining to thermal inertia are more subtle; statistical mechanical theories have been called upon, including the maximum entropy formalism [51, 52], projection operators, [52], and even some

entropy-less theories [53], see reviews [54–56]. Thus, while theories exist of the effect in the entropy representation, its physical picture and objectivity when passing to other frames remains to be worked out. This motivates our study of thermal inertia in the energy representation of thermodynamics.

A recently developed variational approach to irreversible fluids [57–59] is an efficient tool to investigate the thermal inertia systematically and quantitatively, so long as some consequences from kinetic theory can be incorporated into the Lagrangian or Hamiltonian functions of the fluid, L and H respectively. The approach, which extends the “reversible” descriptions of fluids [60] yields systematically all the effects regarded standardly as “irreversible”. It does this by appending the entropy flux $\mathbf{J}_s = \rho_s \mathbf{u}_s$ and its conjugate momentum-like variable (“thermal momentum”, $\partial L / \partial \mathbf{u}_s$) to the set of classical variables, the densities of mass and entropy, ρ and ρ_s and their conjugates. (The conjugates are of course not independent because of their dependence on the field variables.) According to Muschik [61], the field densities ρ_s and \mathbf{J}_s can be taken as primitive variables; the entropy transfer velocity $\mathbf{u}_s = \mathbf{J}_s / \rho_s$. The use of \mathbf{J}_s , augmenting the classical variables, suffices to complete the variational theory of reversible fluids; the additional inclusion of certain dissipation potentials enables one to develop the theory of usual irreversible fluids. Our restriction to fluids is due to the neglect of the elasticity effects in the analysis.

In the field formulation including the entropy flux, the second law constraint or Clausius-Duhem equation has the usual form, equating the four-divergence of the entropy four-flux $(\rho_s, \rho_s \mathbf{u}_s)$ to the entropy source σ_s . The fundamental nature of such a balance law, suggested by various H theorems as the positiveness of σ_s , can be and had been disputed in both kinetic theory [62] and information-theoretic approaches [52]. In the Chapman-Enskog expansion of σ_s a definite sign can be attached only to the first order $\sigma_s^{(1)}$, but not to $\sigma_s^{(2)}$ or any of the higher order terms. (Yet the fact that it is possible to imagine fields that make $\sigma_s^{(2)}$ negative, is no evidence that in physical situations the sum of $\sigma_s^{(2)}$ and higher order terms could be both negative and larger in magnitude than $\sigma_s^{(1)}$.) In Grad’s expansions the neglect of higher order moments is accepted, with no assurance that the expansions converge in general; the difficulty of this sort is also observed in informationtheoretic approaches [52]. (See however Eu’s modified moments [7]). Both Grad’s kinetic results [1] and Nettleton’s phase-space integral [52] show that the entropy flux is not exactly equal to \mathbf{j}_q/T . Thus there is no certainty that the field balance equation for the entropy is a universal law in systems other than dilute gases. It is rather the concavity of the entropy with respect to all its variables, including derivatives, which is regarded as a strongest formulation of the second law [41, 62, 63]. However, a universality of the entropy balance built in the action

functional is by no means necessary [59, 60]. Indeed, in the action formalism, the source sign properties in the balances of the entropy or mass are inessential for the development of the fundamental equation of Gibbs in the sense of Callen [41]. They influence only particular results, e.g. positivity of the partial derivative of the energy E with respect to the so-called thermal phase η , which is equal to σ_s [59]. What is essential is rather a proper set of constraints built into the action integral (leading to the pressure) and the functional structure of σ_s .

In particular, it is inessential whether or not σ_s is positive in any case, or, as one may presuppose [62], only up to the first order terms. When chemical reactions occur, the mass sources of individual species *can* obviously be of either sign. Yet, in entropy balances thus far investigated, we assumed a standard, positive entropy source, σ_s , evaluated on the basis of the experimentally-supported Joule-Onsager expression. This implies that a thermal mass must be created when the entropy and the thermal inertia are equivalent, which is the case of a constant θ , *the thermal inertia per unit of entropy*. This conclusion is a subsidiary, particular result of the Joule-type expression for a positive σ_s . More generally, the concept of a thermal mass arises from breaking the total mass, a conserved quantity, into two components: a “bare” or kinematic mass associated with material flow, and a thermal mass associated with conductive flow of heat and entropy. Because entropy and mass transport change with time and position, these components of the total mass are not conserved individually.

However, since the constancy of θ is only an approximate result of the Grad’s theory and the entropy source sign does not affect the approach, one may speculate whether the constraint on the thermal mass may be more relevant than on the entropy. This is actually equivalent to the hypothesis that the thermal mass is a basic entity, and its source σ'_s can replace the entropy source σ_s . This statement may seem drastic, but it is actually not. The basic constraint in any action formulation for an equilibrium fluid is that of mass balance because the Euler equation and the tensor of matter emerge the same whether or not the entropy s is changed in the action Λ . In the irreversible case, with a source, the effectiveness of the entropy constraint is not well established; there is no proof that a thermal mass constraint with a source must be less effective than standard mass constraints. Quite generally balances of nonconserved mass (in chemical and nuclear reactions) are at least as relevant as entropy balance constraints.

To achieve our goals we bring into the context of the thermal mass the field Lagrangian Λ which we previously used in treating nonequilibrium entropy. Here, it incorporates all balances of mass and identity with corresponding sources. Both Lagrangian and Hamiltonian formalisms can be used, as in traditional frames [59, 64]. As soon as the microscopic information is incorporated into Λ or H , further analysis can be purely macroscopic, so that nontrivial, macroscopic definitions

of heat, diffusion and viscous work follow as *nonequilibrium* effects caused by the transport of the thermal mass in the fluid frame. The energy density E arises as a derivative-dependent quantity, or a sort of pseudo-thermodynamic variable. Yet, one may show that the change of the representation, e.g., to that of entropy, makes each new potential such a variable; this is an effect of different equilibrium reference states in various representations. The related thermodynamic representations are, however, transformable from one to another, demonstrating the equivalence between diverse representations of the same nonequilibrium state [58].

A part of the paper is devoted to analysis of two kinds of hypotheses linking the thermal mass with the entropy. The hypothesis postulating a finite constant θ or the equivalence between the thermal mass and the entropy is the simplest; it preserves the Clausius-Duhem inequality and main features of the usual entropic description in an easiest way. Yet it has an advantage of direct evaluation of the energy of the entropy flow as the kinetic energy of the thermal mass. On the other hand, it is through the hypotheses that make use of variable θ that new dimensions can be added to our understanding of thermodynamics. With this belief we analyze expressions relating the state of the system to the coefficient $\theta = \rho'_s/\rho$, the thermal mass per unit of the entropy. Our discussion examines the quantitative implications of both of these hypotheses, concerning what fraction of the total mass of a fluid should be attributed to its thermal mass. Some large differences appear. While these should perhaps not be surprising, we want to stress that, due to the approximate nature of the Grad's approach, even results from his relatively reliable model may be only approximate (even if we do have the proof that they can always obey the stringent conditions from equilibrium thermodynamics that link energy and pressure, Section 9).

Our main result is a nonlinear theory of the thermal inertia in the energy representation. We show that it is possible to quantify the inertial effects, organize their systematic analysis, interpret them physically and compare a few of the available hypotheses. In contrast to refs. [58] and [59] where analytical aspects of the fluid field theory in the classical frames prevail, the physical reasoning which deals with a thermal-mass frame is no less essential than the formal analysis. A simple field symmetry leads to a new conclusion so far undiscovered in the extended irreversible thermodynamics (EIT): for any positive σ_s (Clausius-Duhem range), a finite thermal inertia, stemming from a finite thermal momentum, causes necessarily creation of what we call the thermal mass and disappearance of the so-called bare mass. Our explicit treatment of the thermal mass should not be confused with other approaches to the (dual) inertial effects in the entropy representation, which can occasionally be related to the extremal behavior of the entropy [52, 65].

2. Definitions of Thermal Momenta, Bare Mass and Thermal Phase

For simplicity we restrict our discussion to systems of a single component or species. Through the rest of this text it is assumed that the reader is familiar with our earlier analyses of the traditional frame of the variables ρ , ρ_s , \mathbf{u} , \mathbf{u}_s [58, 59]. As the information gained so far shows [57–59, 66], there is no single obvious definition to prefer for “thermal momentum”. Rather, there are various thermal momenta, at least from the formal viewpoint, defined as partial derivatives of a macroscopic Lagrangian (or Lagrangian density) L with respect to the macroscopic velocity of the entropy transfer, $\rho_s = \partial L / \partial \mathbf{u}_s$ (flux \mathbf{J}_s and $\mathbf{u}_s = \mathbf{J}_s / \rho_s$), carried out at a constant velocity of the matter and constant values of whatever densities are the thermodynamic variables of L . One can also differentiate L with respect to various diffusion velocities $\mathbf{v}_s = \mathbf{u}_s - \mathbf{u}$. One then obtains the corresponding thermal momenta of diffusion.

The values of any such partial derivative depend very much on which remaining thermodynamic variables of L are kept constant. The thermal momenta so obtained actually constitute a sort of generalized momenta which depend on the set of variables selected, as with the generalized momenta of mechanics. For the development of a concept of thermal mass, it is essential that some special or “canonical” momenta and corresponding densities (ρ_m and ρ'_s) exist and that the effect of these densities on the momenta can be separated in the reversible case, for $L = L^0$. Since L^0 has no explicit matter-field interaction term, only inherent inertial terms affect these momenta, which are then the physical, kinetic momenta, as distinguished from any generalized (canonical) momenta which may be dependent on other factors.

Because of their separable kinetic momenta in the diagonal kinetic energy, the canonical densities represent the distinctive physical entities, which yield the “physical” momenta related to inertial behavior of these entities. As our diagonalizing of the kinetic part of the Lagrangian L^0 (Grad’s kinetic energy K) in Section 6 shows, a state function $\theta(\rho, \rho_s)$ exists such that the densities of the thermal mass, $\rho'_s = \rho_s \theta$, and the so-called bare mass, $\rho_m = \rho - \rho'_s$, can indeed be separated in K . The thermal mass and the bare mass are then the “canonical entities” and the partial derivative of L^0 with respect to the absolute transfer velocity \mathbf{u}_s becomes the absolute, “physical” thermal momentum (that of L^0 with respect to the relative velocity $\mathbf{u}_s - \mathbf{u}$ become the physical thermal momentum of diffusion). Only after diagonalizing of K is completed one can find the density of the thermal mass, ρ'_s , as the ratio of $p_s = \partial L^0 / \partial \mathbf{u}_s$ to u_s , where u_s and p_s are the absolute values of \mathbf{u}_s and \mathbf{p}_s . Hence the fraction attributed to the thermal mass in the total, observable mass of the fluid which has the density ρ equals $f = \rho'_s / \rho$. For the fluid composed of the particles of mass m , the thermal mass per single particle

is $m_s = fm = \rho'_s m / \rho = \theta sm$, where s is the specific entropy and θ equals ρ'_s / ρ_s , the thermal mass per unit of the entropy. The units of ρ_s and ρ'_s are respectively the entropy density and the mass density; for any model of a constant θ the density ρ'_s is simply the entropy density expressed in mass units.

When the fraction f of the total mass attributed to the thermal mass is known, the supplementary fraction $(1 - f)$ may be called the “bare mass fraction” of the medium. The properties of the bare mass are here designated with the subscript m . Since the sum of the (“partial”) momentum density of the bare mass and that of the thermal mass yield the total momentum density or the mass flow, $\mathbf{p}_s + \mathbf{p}_m = \rho \mathbf{u}$, (Sect. 11), the total flow can be written in terms of the thermal mass and bare mass properties as $\rho \mathbf{u} = \rho'_s \mathbf{u}_s + \rho_m \mathbf{u}_m$, where $\rho = \rho'_s + \rho_m$. This means that the hydrodynamic velocity, an observable, acquires the definition $\mathbf{u} = f \mathbf{u}_s + (1 - f) \mathbf{u}_m$ using the fraction of the total mass attributed to the thermal mass in the fluid. The hydrodynamic velocity is, of course, the mean velocity of the center of mass whereas $\mathbf{u}_s - \mathbf{u}$ and $\mathbf{u}_m - \mathbf{u}$ represent the diffusion velocities of the thermal mass and the bare mass in the barycentric frame. The entropy flux $\mathbf{j}_s = \rho_s (\mathbf{u}_s - \mathbf{u})$ is related to the heat flux $\mathbf{j}_q = T \mathbf{j}_s$; the bare mass flux, to the barycentric self-diffusion. This explains equations (1) and (2) used in analysis which follows, as the relevant definitions. Since there is no assurance of the equivalence of the thermal mass and the entropy, the index s labelling the thermal mass-related quantities should be understood as “a quantity related to the entropy” rather than “the quantity uniquely determined by the entropy”.

In the action functional A , the Lagrange multiplier, η , of the traditional entropy balance is the velocity potential of irrotational entropy paths and, at the same time, the Lagrangian action of the entropy transferred [59, 60, 62]. It has been postulated that η , called briefly the thermal phase, is relevant to describe irreversible processes via an action formalism. Its thermal-mass-frame counterpart is designated here by η' ; the thermal mass multipliers are related to the corresponding entropy multipliers, modulo the factor θ^{-1} . Exploiting our earlier conclusion that the balances of nonconserved mass are at least as relevant as the entropy balance constraint (also supported by limitations on the basic nature of Clausius-Duhem inequality), we have adopted here the approach which imposes that constraint on the thermal mass balance rather than on the entropy. Our associated formal argument is a selfconsistent nonlinear analysis (Sects. 5 and 6) which shows that the traditional frame using the entropy (“entropy frame”) and the thermal mass frame can always be transformed one into another. This leads to the second law in a particular form, where, thanks to the nonlinear transformations, eqs. (11) and (22–25), it manifests itself through the source of the thermal mass rather than the entropy. The equivalence of the two descriptions follows from the invariance of the variational principle in various representations, in particular in the classical representation

with the entropy source and a “canonical representation” with the source of the thermal mass.

3. Towards Compatibility with Classical Kinetic Energy

Let us summarize the consequences of the finite thermal momentum. The bare momentum density of the matter, \mathbf{p}_m , is the total momentum density, with the thermal part \mathbf{p}_s subtracted. The density \mathbf{p}_m is such that $\mathbf{p}_m = \rho\mathbf{u} - \mathbf{p}_s$ is associated with a bare mass density $\rho_m = \mathbf{p}_m/\mathbf{u}_m$, where the bare mass properties ρ_m and \mathbf{u}_m obey the transformations

$$\rho = \theta\rho_s + \rho_m = \rho_m, \quad (1)$$

$$\rho\mathbf{u} = \theta\rho_s\mathbf{u}_s + \rho_m\mathbf{u}_m = \rho'_s\mathbf{u}_s + \rho_m\mathbf{u}_m. \quad (2)$$

The multiplication of ρ_s by θ transforms the entropy density into mass units. Various hypotheses may lead either to θ constant or to θ being a function of state. Having $\rho'_s = \theta\rho_s$ enables one to treat the thermal mass as the zeroth component, with an effective mass density ρ'_s . In this work we search for a hypothetical “equation of state” linking ρ'_s with classical variables ρ_s and ρ ,

$$\rho'_s = \theta(\rho_s, \rho)\rho_s, \quad (3)$$

and consistent with the kinetic energy K having “diagonal” structure:

$$K = \frac{1}{2}\theta\rho_s\mathbf{u}_s^2 + \frac{1}{2}\rho_m\mathbf{u}_m^2 = \frac{1}{2}\rho'_s\mathbf{u}_s^2 + \frac{1}{2}\rho_m\mathbf{u}_m^2. \quad (4)$$

Within the thermal mass context, the form (4) can be justified by relativistic theories for low velocities [67, 68]. We determine the state function $\theta(\rho, \rho_s)$ in Sect. 6 and show that for the Grad’s solution of Boltzmann equation it has the form of eq. (19), provided of course that no limitation is imposed on the constancy of θ . We also show that while $\theta(\rho, \rho_s)$ is consistent with eq. (4), the difficulties arising from its variability lead to applying θ as a pseudo-constant multiplier of the entropy source instead of using a rigorous transformation, eq. (11). Otherwise, we develop a nonlinear theory of the state transformation from the entropy frame to the thermal mass frame, still preserving the well-established thermostatics and hydrodynamics. We observe that the “natural” variables of the internal energy are changed by the finiteness of θ . Yet, one still obtains an equivalent thermophysics with the same tensor of matter (Sects. 9 – 11).

4. Trivial Intrinsic Symmetry: Conservation of Total Mass Versus Nonconservation of Thermal Mass

The passage from the conventional frame (thermodynamic or entropy frame) to the frame of thermal mass involves not only transformation of ρ_s into $\rho'_s = \rho_s \theta(\rho_s, \rho)$ but also the replacement of the total mass density ρ by the density of the bare mass $\rho_m = \rho - \rho'_s$; this is nontrivial even if θ is constant. Applying our entropy-frame technique [59, 68] to construct a kinetic potential L for processes with sources, an irreversible L of the thermal mass frame is

$$L = K(\rho'_s, \rho_m, \mathbf{u}_s, \mathbf{u}_m) - \rho_e(\rho'_s, \rho_m) - \eta'_s \sigma'_s - \phi_m \sigma_m, \quad (5)$$

where $K(\rho'_s, \rho_m, \mathbf{u}_s, \mathbf{u}_m)$ is the diagonal kinetic energy, eq. (4). The phase variables η'_s and ϕ_m , the Lagrange multipliers of the related mass balances, eqs. (8) and (9), are the thermodynamic adjoints of sources σ'_s and σ_m .

The total mass must of course be conserved, so that any change of the thermal mass σ'_s must be compensated by that of the bare mass, σ_m . This effect manifests itself as a trivial, global symmetry of the action A : namely, A should be invariant with respect to a common shift of all phases (η'_s and ϕ_m) by the same quantity δ ,

$$\eta'_s \rightarrow \eta'_s + \delta \quad \text{and} \quad \phi_m \rightarrow \phi_m + \delta. \quad (6)$$

This leads to the invariance condition of L with respect to eq. (6) in the form

$$-\delta(\sigma'_s + \sigma_m) = 0 \quad (7)$$

or $\sigma'_s = -\sigma_m$ that any source of thermal mass must be compensated by a sink of equal strength. An alternate form of (7) is the global equation of continuity (10). It follows when one adds the extremum conditions of the action A with respect to the phases ϕ_m and η'_s . This action uses L of eq. (5) and contains the product of the total time derivatives of the phases and densities as in eq. (31). One thus obtains the balance equations for the thermal mass and bare mass

$$\frac{\partial \rho'_s}{\partial t} + \nabla \cdot (\rho'_s \mathbf{u}_s) = \sigma'_s, \quad (8)$$

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{u}_m) = \sigma_m. \quad (9)$$

Since the symmetry of A imposes $\sigma'_s = -\sigma_m$, from eqs. (1) and (2) the sum of the eqs. (8) and (9) is sourceless, so the total mass is indeed conserved:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (10)$$

Thus global mass conservation may be regarded as a consequence of the trivial symmetry of the action with respect to the simultaneous shift of all the phases. The purpose of this simple analysis is to stress that the conditions (7), (10), and (6) with $dA = 0$ are all equivalent, regardless of which is used.

5. Hypotheses Relating Conversion of Bare Mass into Thermal Mass

Eq. (8) can strictly be equivalent to the standard entropy balance only if θ is a constant. This observation is consistent with the one or few hypotheses associated with de Broglie microthermodynamics [69–71] which link the entropy generation with the creation of thermal mass (entropy) within his “piloting wave”, and the consequent necessity of attenuating the bare mass. The case with a constant θ can also be related to Veinik’s [72, 73] hypothesis of elementary thermal quanta, particles of discrete entropy, which he measures in terms of the entropy of a single photon of the black body radiation. The entropy of blackbody photons is normally taken to be a state-independent constant per photon, see, e.g., Landau [74]. Away from blackbody equilibrium, when the isotropy of photons may be lost and the chemical potential is nonvanishing [75], the entropy per single photon ceases to be a constant, but Veinik’s measure is still the black photon entropy. Different choices among these hypotheses may give differing numerical values of θ , and different physical interpretations of the equivalence of the thermal matter with the entropy, but they lead to the same formal model. However θ is a variable function rather than a constant for the Grad-Boltzmann model; which makes it differ from the others.

For the hypothesis based on the Grad’s approach the source of thermal mass, σ'_s , is not equivalent to that of the entropy, cf. eq. (11). Since the real changes of state are seldom large, an averaged $\bar{\theta}$ instead of the instantaneous θ can be applied with some care. Yet, in order to preserve the strictly equivalent pictures, it will be necessary to apply an approach involving nonlinear transformations (eqs. (11) and (25) – (28), for Grad’s model) within which the entropy picture and the thermal mass picture remain two different, complementary formalisms describing the Clausius-Duhem theory.

Using $\rho'_s = \theta(\rho_s, \rho)\rho_s$ in eq. (8) yields in the traditional frame

$$\begin{aligned} \sigma_s = \theta^{-1} \left\{ \sigma'_s(\rho, \rho_s, \nabla\rho, \nabla\rho_s, \mathbf{u}, \mathbf{u}_s) - \rho_s \left[\frac{\partial\theta}{\partial\rho_s} \left(\frac{\partial\rho_s}{\partial t} + \mathbf{u}_s \cdot \nabla\rho_s \right) + \right. \right. \\ \left. \left. + \frac{\partial\theta}{\partial\rho} \left(\frac{\partial\rho}{\partial t} + \mathbf{u}_s \cdot \nabla\rho \right) \right] \right\}, \end{aligned} \quad (11)$$

where σ'_s is the known thermal mass source transformed from the canonical frame of the mass variables $(\rho'_s, \rho_m, \mathbf{u}_s, \mathbf{u}_m)$ to the conventional frame of the entropy

density $(\rho_s, \rho, \mathbf{u}_s, \mathbf{u})$. Inversely, in the canonical frame

$$\begin{aligned} \sigma'_s &= \theta \sigma_s + \rho_s \left(\frac{\partial \theta}{\partial t} + \mathbf{u}_s \cdot \nabla \theta \right) = \theta \sigma'_s(\rho_m, \rho'_s, \nabla \rho_m, \nabla \rho'_s, \mathbf{u}_m, \mathbf{u}_s) + \\ &+ \theta^{-1} \rho'_s \left[\frac{\partial \theta}{\partial \rho'_s} \left(\frac{\partial \rho'_s}{\partial t} + \mathbf{u}_s \cdot \nabla \rho'_s \right) + \frac{\partial \theta}{\partial \rho_m} \left(\frac{\partial \rho_m}{\partial t} + \mathbf{u}_s \cdot \nabla \rho_m \right) \right], \end{aligned} \quad (11a)$$

where σ_s is the known entropy source (e.g., the sum of the two dissipation functions, $\Phi + \Psi$) transformed from the conventional frame to the canonical frame. Eq. (11a) determines $\sigma_s(\rho_s, \rho, \mathbf{u}_s, \mathbf{u})$ from $\sigma'_s(\rho'_s, \rho_m, \mathbf{u}_s, \mathbf{u}_m)$ known as the mass-frame solution of the variational problem involving eq. (5). Conversely, eq. (11a) yields $\sigma'_s(\rho'_s, \rho_m, \mathbf{u}_s, \mathbf{u}_m)$ from $\sigma_s(\rho_s, \rho, \mathbf{u}_s, \mathbf{u})$ known as the variational solution in the conventional frame. These formulae are exact, so long as the relaxation times of the system are fast enough that the state functions exist. (An independent way of constructing the dissipative potentials in a definite frame is through squaring equations of dissipative kinetics [65].) Any changes (or simplifications) in eqs. (11) or (11a) imply relaxing the equivalence requirement between the two formalisms. These changes can nonetheless be of interest as leading to possibly improved, independent theory.

The three hypotheses for the thermal mass are different from the physical viewpoint. The constant- θ hypotheses, which operate with L having any instantaneous θ in eq. (4), are perhaps the most elegant from the esthetic viewpoint but they are physically very exotic. They imply the entropy source is associated with the “dissolving-like” or “shrinking-like” behavior and ultimate disappearance of the bare matter into a “bath of indefiniteness” occupied by the thermal mass, identified here with the entropy, implying constant θ . This is consistent with the de Broglie hypothesis which identifies the entropy apart from the elementary particle; rather, it is attributed to the surrounding bath or to its “piloting wave” as a “subquantum” environment. De Broglie treats his θc^2 as an intrinsic temperature of micro-objects and atoms at rest. Constant- θ hypotheses are also compatible with Veinik’s idea of heat particles (particles of a thermal matter) equivalent in some sense to the entropy. These hypotheses postulate in fact an equivalence principle between entropy and matter characterized by a universal constant θ .

On the other hand, the third hypothesis, based on the nontrivial constitutive equation, $\rho'_s = \theta(\rho_s, \rho) \rho_s$, consistent with the Grad’s approach, disallows any serious identification of the thermal mass with the entropy while still admitting a link between these two entities. As eq. (19) shows, Grad’s function $\theta(\rho_s, \rho)$ grows with the specific entropy s or temperature T in the range of low s , but it decreases with s or T at constant ρ in the range of high s . For the ideal hard-sphere gas, $\theta(s)$ is presented on Fig. 1. Thus, the link between the thermal mass and the entropy in the Grad-Boltzmann model is weaker than in the two previous hypotheses, as

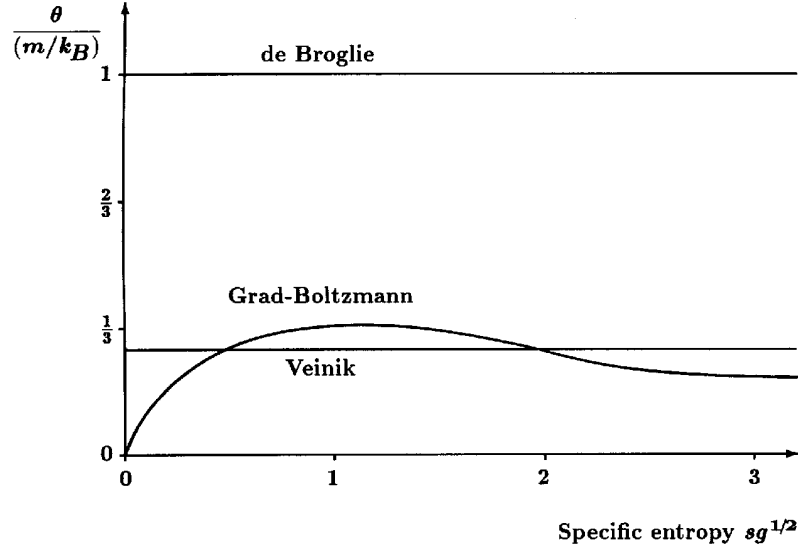


Fig. 1. Comparison between inertial coefficients θ for various hypotheses of thermal inertia

the positivity of the source σ'_s is not assured by this model. Grad's thermal mass seems thus to be a different entity than the entropy, yet still fundamental as that quantity explaining and quantifying the inertia of the heat flow. To preserve a finite entropy source, this thermal mass must be nonconservative so long as a significant plateau region surrounding the maximum of θ in Fig. 1 excludes any compensation of σ_s by the effect of $\rho_s d\theta/dt$ (eq. 11) in a finite regime. Whenever σ_s is positive, the thermal mass must be created at least in the plateau region of the Grad-Boltzmann θ . Therefore the most legitimate statement regarding this model at the present time is that already made for the models of constant θ : it implies a *nonconservative* thermal mass.

The power of any variational formulation lies in its applicability to arbitrary nonlinear transformations. Hence any cases of variable θ are manageable by the theory if only appropriate data are known. However, since our present knowledge of the dissipation expressions is only approximate, in the presence of sources the equivalence of the two descriptions is as a rule met only approximately, associated with applying θ as a pseudo-constant. The frictional nature of the mass-frame dissipation and results obtained from linear analyses may help us to work out effective models for σ'_s . Until this problem is resolved, one must be content with only approximate expressions of the averaged type, obeying $\sigma'_s = \bar{\theta}\sigma_s$. It is the plateau on Fig. 1 which allows assumption of constant θ for Grad's model over

quite a broad range of the term $sg^{1/2}$ (surroundings of the extremum point $\xi = sg^{1/2} = 1$).

6. Evaluation of Inertial Function $\theta(\rho_s, \rho)$ from Grad's Approach

In order to find $\theta(\rho_s, \rho)$ compatible with Grad's solution, the diagonal structure (5) is transformed to operate with the conventional variables ρ, \mathbf{u}, ρ_s and \mathbf{u}_s . Substituting into K , eq. (4), the bare mass velocity \mathbf{u}_m expressed as

$$\mathbf{u}_m = (\rho\mathbf{u} - \rho'_s\mathbf{u}_s)(\rho - \rho'_s)^{-1} \quad (12)$$

yields

$$K = \frac{1}{2}\rho'_s\mathbf{u}_s^2 + \frac{1}{2}(\rho - \rho'_s)^{-1}(\rho^2\mathbf{u}^2 - 2\rho\rho'_s\mathbf{u}\mathbf{u}_s + \rho_s'^2\mathbf{u}_s^2) \quad (13)$$

which leads to the quadratic form

$$K = \frac{1}{2}\rho\rho'_s(\rho - \rho'_s)^{-1}\mathbf{u}_s^2 + \frac{1}{2}(\rho - \rho'_s)^{-1}\rho^2\mathbf{u}^2 - \rho\rho'_s(\rho - \rho'_s)^{-1}\mathbf{u}\mathbf{u}_s. \quad (14)$$

After defining the density of the hydrodynamic kinetic energy $K_0 = 1/2\rho\mathbf{u}^2$, eq. (14) can be transformed into a form separating the hydrodynamic and diffusional components of K

$$K = \frac{1}{2}\rho\mathbf{u}^2 + \frac{1}{2}\rho\rho'_s(\rho - \rho'_s)^{-1}\mathbf{u}_s^2 + \frac{1}{2}(\rho - \rho'_s)^{-1}\rho'_s\rho\mathbf{u}^2 - \rho\rho'_s(\rho - \rho'_s)^{-1}\mathbf{u}\mathbf{u}_s. \quad (15)$$

The nonequilibrium contribution to the kinetic energy contained in eq. (15) includes the square of the relative difference of velocities

$$k_\theta = \frac{1}{2}(\rho - \rho'_s)^{-1}\rho'_s\rho(\mathbf{u}_s - \mathbf{u})^2 \quad (16)$$

or, when written in terms of the conventional variables

$$k_\theta = \frac{1}{2}(\rho - \theta(\rho, \rho_s)\rho_s)^{-1}\theta(\rho, \rho_s)\rho_s\rho(\mathbf{u}_s - \mathbf{u})^2. \quad (17)$$

This is of course the same structure as the result for k_g known from the nonequilibrium statistical mechanics of heat transfer (Grad's [1] solution of the Boltzmann equation in the variables ρ, \mathbf{u}, ρ_s and \mathbf{u}_s , [58])

$$k_g = \frac{1}{2}\rho_s^2\rho^{-1}g(\rho, \rho_s)(\mathbf{u}_s - \mathbf{u})^2 \quad (18)$$

where g is the inertial coefficient in Grad's expression for the nonequilibrium internal energy. Under the relaxation time approximation of the Boltzmann equation,

$g = 2/5m^2/k_B^2$ for the ideal hard-sphere gas. By deriving eq. (17) we have shown that the kinetic energy of the Grad fluid is diagonal in the variables ρ_m and $\rho'_s = \theta\rho_s$ for θ obeying eq. (19). This proves that ρ_m and ρ'_s are canonical variables in which the kinetic energy is as simple as it can be. By orthogonal transformations of the spatial coordinates in eq. (4), other canonical densities and velocities could be constructed but then the simple physical interpretation of the variables $(\rho_m, \rho'_s, \mathbf{u}_m, \mathbf{u}_s)$ would be lost. Therefore we restrict ourselves to these canonical variables. Eqs. (17) and (18) differ only by the functions $g(\rho, \rho_s)$ and $\theta(\rho, \rho_s)$ which they use. On comparing (17) and (18) the relation between these functions can be found as

$$\theta(\rho, \rho_s) = \frac{\rho_s \rho g(\rho_s, \rho)}{\rho^2 + g(\rho_s, \rho) \rho_s^2} = \frac{g(s, \rho) s}{1 + g(s, \rho) s^2} \quad (19)$$

or

$$g(\rho, \rho_s) = \frac{\rho^2 \theta(\rho_s, \rho)}{\rho_s (\rho - \theta(\rho_s, \rho) \rho_s)} = \frac{\theta(s, \rho)}{(1 - \theta(s, \rho) s) s} \quad (20)$$

Note that both forms become implicit in the “canonical” variables ρ_s , and ρ_m , when g is variable, the fact which determined the direction of our transformations. However, when g is constant (ideal gas) an explicit form for θ in terms of canonical variables easily follows,

$$\theta(\rho_m, \rho'_s) = (g \rho'_s \rho_m)^{1/2} (\rho'_s + \rho_m)^{-1}, \quad (21)$$

corresponding with the functions

$$\rho_s(\rho'_s, \rho_m) = (g \rho_m / \rho'_s)^{-1/2} (\rho'_s + \rho_m) \quad \text{and} \quad \rho(\rho'_s, \rho_m) = \rho'_s + \rho_m.$$

Now we can collect the relevant formulae and summarize the results. We make the following nonlinear transformation of the canonical variables $(\rho'_s, \rho_m, \mathbf{u}'_s, \text{ and } \mathbf{u}_m)$, in which the total kinetic energy density is diagonal, into the traditional physical variables $(\rho_s, \rho, \mathbf{u}_s \text{ and } \mathbf{u})$:

$$\rho'_s = \frac{\rho_s^2 \rho g(\rho_s, \rho)}{\rho^2 + g(\rho_s, \rho) \rho_s^2} \quad (= \rho_s \theta(\rho_s, \rho)), \quad (22)$$

$$\rho_m = \rho - \frac{\rho_s^2 \rho g(\rho_s, \rho)}{\rho^2 + g(\rho_s, \rho) \rho_s^2} \quad (= \rho - \rho_s \theta(\rho_s, \rho)), \quad (23)$$

$$\mathbf{u}_m = \left(\rho \mathbf{u} - \frac{\mathbf{u}_s \rho_s^2 \rho g(\rho_s, \rho)}{\rho^2 + g(\rho_s, \rho) \rho_s^2} \right) / \left(\rho - \frac{\mathbf{u}_s \rho_s^2 \rho g(\rho_s, \rho)}{\rho^2 + g(\rho_s, \rho) \rho_s^2} \right), \quad (24)$$

$$\mathbf{u}'_s = \mathbf{u}_s. \quad (25)$$

This transformation takes the canonical kinetic potential (5), with its diagonal K , into the kinetic potential of Grad’s theory or the sum of the hydrodynamic kinetic

energy $K_0 = \rho \mathbf{u}^2/2$ and the nonequilibrium energy $k_\theta(\rho_s, \rho, \mathbf{u}_s, \mathbf{u})$, eq. (17), for θ given by eq. (19), thus leading to Grad's eq. (18). See Sect. 10 for the transformation of the corresponding velocity potentials (phases).

With this result, one can investigate the extremum conditions for the simplest K in the canonical thermal mass frame and then pass to the conventional variables (entropy frame) or conversely. We observe that with nonzero θ or g the usual thermodynamic variables are not the canonical variables of the kinetic energy, although they still are the most popular variables of the internal energy. While the results with the Grad approach and hence the specific transformations are approximate, the methodological bonus stemming from K being diagonal is consequential because of the ease of embedding the canonical L 's into various complex physical contexts, when, e.g., generalizing them to the relativistic L 's or quantizing the related Hamiltonians. A reversible relativistic theory of the thermal inertia has been constructed in our complementary paper [78].

7. Behavior of θ Under Various Hypotheses

Once the relation between the functions $g(\rho, \rho_s)$ and $\theta(\rho, \rho_s)$, eqs. (19) and (20), is known, $\theta(\rho, \rho_s)$ can be used to estimate the order of magnitude of the inertial effects. The function $\theta(\rho, \rho_s)$ of an ideal hard-sphere gas depends actually on the single variable $s = \rho_s/\rho$, or the dimensionless variable $\xi = sg^{1/2}$, Fig. 1. It exhibits a maximum at s obeying the condition $(\partial\theta(\rho, s)/\partial s)_\rho = 0$. Fig. 1 depicts properties of θ corresponding to various hypotheses. For small s (low T at a constant ρ), the function $\theta(s)$ of Grad's approach is linear; $\theta = gs = (2/5)(m/k_B)^2 s$, for an ideal hard sphere gas with $g = 2m^2/5k_B^2$. For large s the function θ decreases with s ; $\theta(s) = s^{-1}$. Thus, in this model, while the thermal mass goes up with the entropy, the thermal mass per unit of entropy does not, at high entropies. Eq. (19) yields the limiting entropy $\mathcal{S}^* = g^{-1/2} = (5/2)^{1/2} k_B/m$ at which the low- s range terminates. For the ideal gas the maximal $\theta = (1/2)g^{1/2}$, i.e. it equals about $(1/3)m/k_B$. This value of θ defines the maximal thermal mass per unit of entropy, in this model. This maximum can be compared with the de Broglie value $\theta = m/k_B$ [67–71].

A comparison of average values of θ would emphasize the differences since the average values of θ of the Grad approach are about one half of the maximal θ , i.e. they are of the order of $(1/6)m/k_B$. These discrepancies can be ascribed to the nature of de Broglie's hypothesis as his $\theta = m/k_B$ seems merely a limiting quantity corresponding with states of sufficiently high T , when all the observed mass is thermal. (See also the discussion of the fraction of the observed mass assignable as thermal mass in the next section and Fig. 2.) Yet that hypothesis fosters the idea of an equivalence between the entropy and matter which justifies deeper exploration

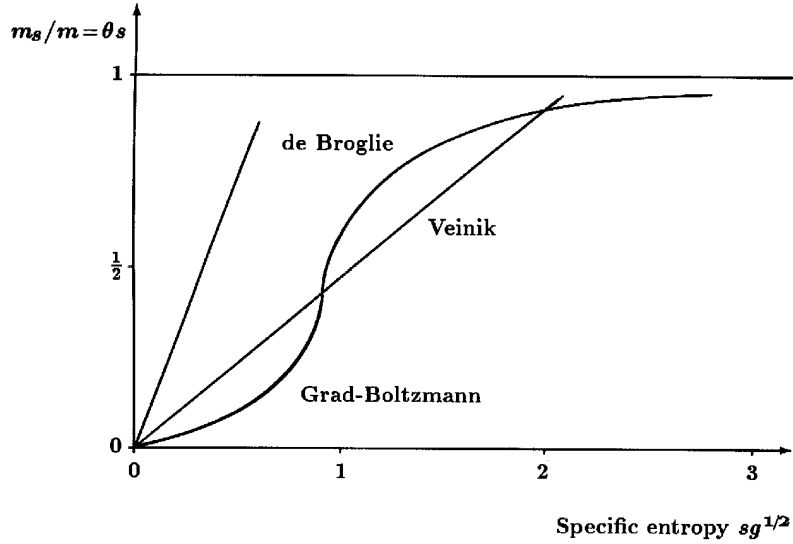


Fig. 2. Fraction of the thermal mass in the total mass for various hypotheses of thermal inertia

because, in that mode, the correspondence between dynamic and static pressures is always unambiguous [67, 68]. Operating with general models of a variable θ , (e.g., a derivative-dependent θ in which the direct correspondence between the two pressures may cause problems), one might use a constant, averaged value for $\theta = \bar{\theta}$, obtained by averaging eq. (19). Such a θ would be the order of $(1/3)m/k_B$ rather than de Broglie's $\theta = m/k_B$.

Nonetheless, a value close to $(1/3)m/k_B$ could be preferred on the basis of a different concept. Were one to accept Veinik's [72, 73] idea that an elementary entropy \mathcal{S}^* exists, one would define the limiting θ as the constant coefficient equal to m/\mathcal{S}^* . Then, with his concept that \mathcal{S}^* should be equal to the (state-independent) entropy of a single photon of the black body radiation, $\mathcal{S}^* = 3.6k_B$, one evaluates θ as equal to $m/\mathcal{S}^* = (1/3.6)m/k_B$ which is close to the maximal value of the Grad-Boltzmann theory.

While a definitive statement regarding a best choice of θ cannot yet be made, there is no doubt now that a nonvanishing θ leads to a more physical picture of the fluid motion than $\theta = 0$, because it eliminates the paradox of infinite propagation speed of thermal disturbances. With this model, the thermal waves propagate with the finite speeds $c_0 = (T/g(\theta)c_p)^{1/2}$, or, for ideal gas, $(k_B T/m)^{1/2}$; hence any values of θ of the order of m/k_B are physically plausible, as the signal speed is

then of the order of the mean thermal velocity. For the role of θ in the definition of nonequilibrium temperature, see [57, 67, 68].

8. Fraction of Total Mass Attributed to the Thermal Mass

The problem of thermal inertia is far from closed. As distinguished from Fig. 1, Fig. 2 depicts the thermal mass per unit of total or measurable mass of the ideal gas, or the fraction $f = m_s/m$. In terms of the macroscopic quantities, $f = \theta\rho_s/\rho = \theta s$. The corresponding fraction of the bare mass is then $1 - m_s/m$ or $1 - \theta s$. The “red herring” of the constant- θ hypotheses is their counterintuitive implication: f can achieve a value of unity for a finite, sufficiently large, specific entropy $s = s^* = \theta^{-1}$ where the whole mass becomes thermal, and then, for s larger than s^* , the fraction attributed to the thermal mass within the total mass becomes larger than unity so the corresponding fraction of bare mass becomes negative. This may suggest that the constant- θ hypotheses break down, or, if one admits an eccentric interpretation, that the particles comprising the bare matter become particles of negative energy. For the de Broglie hypothesis the limiting $s^* = \theta^{-1} = k_B/m$, (limiting $\xi^* = g^{-1/2}k_B/m = 0.632$). For the Veinik hypothesis, s^* is about $3.6k_B/m$ ($\xi^* = g^{1/2}s^*$ is about 2.275). These effects might perhaps be not such a surprise for quantum theories where antiparticles may exist. However, we surely cannot discuss this issue seriously in terms of the simple, classical, phenomenological description used here. Therefore, we only stress that for $\theta s > 1$, or for $s > \theta^{-1}$, or for the dimensionless $\xi > s^*g^{1/2}$, the constant- θ hypotheses must allow one of the entities, the thermal or bare mass, to become negative.

On the other hand, in Grad’s model, the fraction of the total mass assignable as thermal mass is always less than unity. From eq. (19) for an ideal gas

$$\theta k_B/m = \frac{\xi}{1 + \xi^2} \sqrt{\frac{2}{5}}, \quad f = \theta s = \frac{\xi^2}{1 + \xi^2}, \quad (26)$$

where $\xi \equiv g^{1/2}s = (2/5)^{1/2}sm/k_B$. The fraction f obviously approaches the unity in the limit of very large s . Thus, in this model the whole mass becomes the thermal mass, for very large specific entropy. A generalization of these (ideal gas) data is possible via a generalized inertial coefficient g of the real fluids: $g(\rho, \rho_s) = T\rho(c_p G)^{-1}$ [34]. Here, c_p is the specific heat and G is the shear modulus which simplifies to the pressure P in the case of the (ideal) gas. For this ideal case our $g(\rho, \rho_s)$ simplifies to $g = 2/5m^2/k_B^2$. Since the fraction $f = s\theta$, then, using eq. (19) leads to the following expressions for the thermal and bare masses, m_s and m_m , contained in the single particle that has the total mass m :

$$\frac{m_s}{m} = \frac{s^2 T \rho / G c_p}{1 + s^2 T \rho / G c_p}; \quad \frac{m_m}{m} = \frac{1}{1 + s^2 T \rho / G c_p}. \quad (27)$$

They imply that practically the whole mass becomes thermal at sufficiently high T 's and constant volume. These masses (27) correspond to the reversible momenta $m_s \mathbf{u}_s$ and $m_m \mathbf{u}_m$ per particle, whose sum is $m \mathbf{u}$. Their multiplication by the number density, n , yields the hydrodynamic momenta $\rho'_s \mathbf{u}_s$, $\rho_m \mathbf{u}_m$ and $\rho \mathbf{u}$. These momenta and densities are believed to be the physical quantities, whereas the pertinent derivatives of the kinetic potential L^0 with respect to various velocities related to \mathbf{u}_s and \mathbf{u}_m , obtained when arbitrary state variables are kept constant, play the role of the generalized momenta.

In short, we admire the Grad's model of θ because: a) it avoids the strange effect of a negative bare mass at high T , b) it agrees well with experiments of thermal conductivity and c) its transformation equations convert the thermal conductivity model into a natural frictional model of the thermal mass diffusion. From the viewpoint of these criteria the constant- θ models are more restrictive. This does not free us from concerns about this model since its compatibility with statics is not obvious in the range of the variable θ . The corresponding test is performed below.

9. Compatibility with Statics and Invariant Properties of Pressure

Let us compare the Gibbs equation of classical thermostatics with its transformed counterpart dealing with the canonical (mass) variables:

$$\begin{aligned} d\rho_e(\rho_s, \rho) &= T d\rho_s + \mu d\rho = T d(\theta^{-1} \rho'_s) + \mu d(\rho'_s + \rho_m) \\ &= \left(\frac{T}{\theta} + \mu - \frac{T \rho'_s}{\theta^2} \frac{\partial \theta}{\partial \rho'_s} \right) d\rho'_s + \left(\mu - \frac{T \rho'_s}{\theta^2} \frac{\partial \theta}{\partial \rho_m} \right) d\rho_m \\ &\equiv \mu_s d\rho'_s + \mu_m d\rho_m. \end{aligned} \quad (28)$$

This equation stems from the invariance of the internal energy in the two frames. Its second line defines the equilibrium canonical intensities: the chemical potentials of the thermal mass and of the bare mass, μ_s , and μ_m . For the hypotheses of constant θ , μ_m equals the standard chemical potential μ , whereas μ_s equals $T/\theta + \mu$. (For these hypotheses one could also use the "thermal potential" $T' \equiv \theta \mu_s$, a sort of canonical temperature, equal $T + \theta \mu$). In the mass frame the intensities μ_s , and μ_m replace the standard intensities T and μ of the conventional description of equilibrium. As in the entropy frame [59], the mass phase variables η'_s and ϕ_m , the Lagrangian multipliers of the mass balances (8) and (9), do play an explicit role in dissipative processes.

From eq. (21), in the mass frame, Grad's variable θ obeys:

$$\rho'_s \frac{\partial \theta(\rho_m, \rho'_s)}{\partial \rho'_s} + \rho_m \frac{\partial \theta(\rho_m, \rho'_s)}{\partial \rho_m} = 0. \quad (29)$$

Correspondingly, the equilibrium pressure function in this frame, or the appropriate Legendre transform of ρ_e , is unaffected by the variability of θ :

$$\begin{aligned}
P' &= \frac{\partial \rho_e'}{\partial \rho_s'} \rho_s' + \frac{\partial \rho_e'}{\partial \rho_m} \rho_m - \rho_e' = \mu_s \rho_s' + \mu_m \rho_m - \rho_e'(\rho_s', \rho_m) \\
&= \left(\frac{T}{\theta} + \mu - \frac{T \rho_s'}{\theta^2} \frac{\partial \theta}{\partial \rho_s'} \right) \rho_s' + \left(\mu - \frac{T \rho_s'}{\theta^2} \frac{\partial \theta}{\partial \rho_m} \right) \rho_m - \rho_e'(\rho_s', \rho_m) \\
&= \frac{T}{\theta} \rho_s' + \mu(\rho_s' + \rho_m) - \rho_e'(\rho_s', \rho_m) = T \rho_s + \mu \rho - \rho_e(\rho_s, \rho) = P.
\end{aligned} \tag{30}$$

Together with the invariant nature of the internal energy, this finding also proves that the enthalpy h is another invariant. Thus, while the usual intensities T and μ become only auxiliary variables in the mass frame, the statics of the mass frame is compatible with that of the entropy frame.

Moreover, one can extend the result (30) to the dynamic situation by determining the extremum value of the field Lagrangian in the mass frame

$$\begin{aligned}
\Lambda &= K(\rho_s', \rho_m, \mathbf{u}_s, \mathbf{u}_m) - \rho_e(\rho_s', \rho_m) - (\eta_s' - \phi_m) \sigma_s' - \rho_m \left(\frac{\partial \phi_m}{\partial t} + \mathbf{u}_m \cdot \nabla \phi_m \right) + \\
&\quad - \rho_s' \left(\frac{\partial \eta_s'}{\partial t} + \mathbf{u}_s \cdot \nabla \eta_s' \right) - \gamma_m \left(\frac{\partial \lambda_m}{\partial t} + \mathbf{u}_m \cdot \nabla \lambda_m \right) - \gamma_s' \left(\frac{\partial \lambda_s'}{\partial t} + \mathbf{u}_s \cdot \nabla \lambda_s' \right). \tag{31}
\end{aligned}$$

This includes the diagonal kinetic energy (4) and the kinetic potential (5). The equality $\sigma_s' = -\sigma_m$ has been used to eliminate the source of the bare mass from eq. (5) so that now only the thermal mass source is explicit. We take γ and λ , as identity variables which preserve any nonvanishing vorticity of velocity fields. We do not discuss them here since they behave like their entropy-frame counterparts [59].

Substituting into eq. (31) the extremum conditions of Λ in the form of the related Euler-Lagrange equations, one obtains with eq. (28) the extremal Λ in the form of the Legendre transform, generalized from that of eq. (30),

$$\begin{aligned}
\Lambda' &= L - \frac{\delta L}{\delta \rho_s'} \rho_s' - \frac{\delta L}{\delta \rho_m} \rho_m = L + \mu_s \rho_s' + \mu_m \rho_m \\
&= L + \left(\frac{T}{\theta} + \mu - \frac{T \rho_s' \theta \rho_s'}{\theta^2} - \frac{\mathbf{u}_s^2}{2} + \eta' \frac{\delta \sigma_s'}{\delta \rho_s'} \right) \rho_s' + \left(\mu - \frac{T \rho_s' \theta \rho_m}{\theta^2} - \frac{\mathbf{u}_m^2}{2} + \eta' \frac{\delta \sigma_s'}{\delta \rho_m} \right) \rho_m \\
&= \frac{T}{\theta} \rho_s' + \mu(\rho_s' + \rho_m) - \rho_e'(\rho_s', \rho_m) + \eta' \frac{\delta \sigma_s'}{\delta \rho_s'} \rho_s' + \eta' \frac{\delta \sigma_s'}{\delta \rho_m} \rho_m - \eta' \sigma_s' \\
&= T \rho_s + \mu \rho - \rho_e(\rho_s, \rho) - \eta' \sigma_s^{*'} = P - \eta \sigma_s^* = \Lambda.
\end{aligned} \tag{32}$$

Here $\eta' = \eta_s' - \phi_m$ is the relative thermal mass phase and $\sigma_s^{*'}$ is the Legendre transform of the thermal mass source σ_s' . The deviation of the pressure from equilibrium

is the second viscosity effect. Note that the invariance of the product $\eta' \sigma_s^{*l} = \eta \sigma_s^*$ which describes that effect follows directly only for constant- θ hypotheses, whereas the use of eq. (11) is necessary to prove this invariance if θ is variable. In a nondissipative ideal gas σ_s' and σ_s^{*l} vanish and, even when θ varies and the fluid flows, the extremum Λ again equals the equilibrium pressure P . This result is consistent with the analysis of the ideal gas in the entropy frame (cf. eq. (67) of ref. [58] for constant g). The same equilibrium and nonequilibrium pressures can thus be obtained in both frames. These results, which prove the compatibility of Grad's θ with the well-established thermostatic tensor of matter, $\text{diag}(-P, -P, -P, \rho_c)$, are crucial for working with the thermal mass of Grad's model. One may note that analyses in the mass frame are as a rule easier than in the thermodynamic frame.

10. Physics of Phases and Sources in Balance Equations

Because of their crucial role as generators of Legendre transformations, for the thermodynamic compatibility of various frames, transformations of phases (e.g. those of ϕ_m and η_s') associated with those of state are essential. They involve the field Lagrangian Λ , eq. (31), rather than the kinetic potential L , eq. (5). Consider the passage from the canonical frame $(\rho_m, \rho_s', \mathbf{u}_m, \mathbf{u}_s, \phi_m$ and $\eta_s')$, eq. (31), to the entropy frame $(\rho, \rho_s, \mathbf{u}, \mathbf{u}_s$ and $\eta)$. Using eqs. (11) and (12) in eq. (31) yields the Lagrangian of the entropy frame

$$\begin{aligned} \Lambda = & K(\rho_s, \rho, \mathbf{u}_s, \mathbf{u}) - \rho_e(\rho_s, \rho) - \eta \sigma_s - \rho \left(\frac{\partial \phi_m}{\partial t} + \mathbf{u} \cdot \nabla \phi_m \right) + \\ & - \rho_s \left(\frac{\partial \eta}{\partial t} + \mathbf{u}_s \cdot \nabla \eta \right) - \gamma \left(\frac{\partial \lambda_m}{\partial t} + \mathbf{u} \cdot \nabla \lambda_m \right) - \gamma_s \left(\frac{\partial \lambda_s}{\partial t} + \mathbf{u}_s \cdot \nabla \lambda_s \right). \end{aligned} \quad (33)$$

Here $\eta = \theta(\eta_s' - \phi_m)$ is the relative phase of entropy [59], and the entropy source σ_s is related to the thermal mass source σ_s' by eq. (11a). $\lambda_s = \lambda_s' - \lambda_m$ is the initial velocity of the entropy flow and λ_s' itself corresponds to the absolute initial velocity of this flow. Eq. (33), with the identification $\phi_m = \phi$ and $\lambda_m = \lambda$ is the integrand of the entropy-frame action [59] which preserves the usual thermodynamics and the classical description of the heat flow. The production of the thermal mass in the canonical frame leads in the conventional frame to the entropy production. Eq. (33) proves that, if sources are present, it is actually the velocity potential of the bare matter that has been eliminated, ϕ_m , not any "global" matter phase ϕ , whose variation yields the global continuity. This apparent peculiarity is in fact a valid result which amends the interpretation of the velocity potentials ϕ and η .

On the other hand one may describe the bare mass motion with respect to this fluid. Λ contains then the term $-(\phi_m - \eta_s')\sigma_m$ which defines the relative phase of

the bare matter, $\varphi = \phi_m - \eta'_s$ such that $\varphi = -\eta/\theta$. Substituting $\rho'_s = \rho - \rho_m$ and $\mathbf{u}_s = \mathbf{u}_s(\mathbf{u}, \mathbf{u}_m)$ from eq. (2), into eq. (31), one finds

$$\begin{aligned} \Lambda = & K(\rho_m, \rho, \mathbf{u}_m, \mathbf{u}) - \rho_e(\rho_m, \rho) - \varphi\sigma_m - \rho_m \left(\frac{\partial\varphi}{\partial t} + \mathbf{u}_m \cdot \nabla\varphi \right) + \\ & - \rho \left(\frac{\partial\eta'_s}{\partial t} + \mathbf{u} \cdot \nabla\eta'_s \right) - \gamma_m \left(\frac{\partial\lambda}{\partial t} + \mathbf{u}_m \cdot \nabla\lambda \right) - \gamma \left(\frac{\partial\lambda_s}{\partial t} + \mathbf{u} \cdot \nabla\lambda_s \right). \end{aligned} \quad (33a)$$

In this frame, the dissipation manifests itself through the sink of the bare mass; the production of the thermal mass or entropy is only implicit. The equation for global matter conservation follows now from variation of the action A with respect to the absolute phase of thermal mass, η'_s ; otherwise the diminution of the bare mass is governed by the relative phase, φ . From eq. (33a), the phase of the eliminated entity (here that of the thermal mass), η'_s , has again acquired the role played by the total matter phase (ϕ) in a sourceless process, namely, it is the variation of η'_s which yields the global continuity.

Thus, the presence of sources leads to a subtle complication in the interpretation of the velocity potentials (phases) of transferred entities, for both nonconserved and conserved balances. “Bare” entities and their velocity potentials arise naturally, and the relative phases, η or φ , (the differences between the absolute phases and the phase of the eliminated entity) yield balances of bare entities with sources. Otherwise the variation of the phase of the entity that has been eliminated yields the global continuity.

Interestingly, for the constant- θ hypotheses, the equivalence of the entropy and mass with a finite constant θ , identifies the global matter density ρ with a density of a global entropy, $\widetilde{\rho}_s = \rho/\theta = \rho_m/\theta + \rho_s$, which is a conserved property. Conservation of that entropy and the global mass should then be equivalent and this would explain why the variation of the entropy phase η'_s in eq. (33a) yields the global continuity. The thermodynamic entropy should then be the bare, nonconservative entropy, distinguished from the conserved global entropy whose density $\widetilde{\rho}_s$ includes the entropy equivalent of the bare matter. Perhaps the notion of such generalized entropy would explain some paradoxes noted for the dynamic entropies of the contemporary theories [76].

11. Identity of Matter Tensor in Entropy and Mass Frames

The subsequent analysis proves that the two representations yield the same tensor of matter. This, in particular, means that various partial momenta yield the same physical momentum, $\mathbf{p} = \rho\mathbf{u}$ and other components of this tensor. We deal with both the canonical kinetic potential L , eq. (5), and its conventional-frame counterpart, eq. (34). For the two frames compared we display the role of the derivatives $\mathbf{p}_s = \partial L/\partial\mathbf{u}_s$, $\mathbf{p}_m = \partial L/\partial\mathbf{u}_m$, and $\mathbf{p} = \partial L/\partial\mathbf{u}$.

The entropy frame uses L in the variables ρ , ρ_s , \mathbf{u} , \mathbf{u}_s , and η ,

$$L = \frac{1}{2}\rho\mathbf{u}^2 + \frac{1}{2}(\rho_s^2\rho^{-1}g - \eta\rho_s^2\kappa^{-1})(\mathbf{u}_s - \mathbf{u})^2 - \rho_e(\rho_s, \rho) - \rho\psi(\mathbf{x}, t) - \frac{1}{2}\eta\Psi(\rho, \rho_s, \nabla\rho, \nabla\rho_s). \quad (34)$$

It has the Onsager potentials incorporated (κ and Ψ terms of eq. (34); see also eqs. (24) and (28) of ref. [59]). These dissipation potentials can be constructed in a well-established way, via squaring dissipative kinetics [65]. $\theta(\rho, \rho_s)$ of eq. (19) is implicitly present in g of eq. (34). (Note that the regime of a constant θ is not that of an ideal gas for which $g = \text{constant}$ and θ varies as in Fig. 1.)

In the canonical frame, ρ_m , ρ_s , \mathbf{u}_m , \mathbf{u}_s , η'_s and ϕ_m , L takes the form

$$L = \frac{1}{2}\rho_m\mathbf{u}_m^2 + \frac{1}{2}\rho'_s\mathbf{u}_s^2 - \frac{1}{2}(\eta'_s - \phi_m)\theta(\kappa g)^{-1}\rho'_s\rho_m(\mathbf{u}_s - \mathbf{u}_m)^2 + - \rho_e(\rho'_s, \rho_m) - (\rho'_s + \rho_m)\psi(\mathbf{x}, t) - \frac{1}{2}(\eta'_s - \phi_m)\theta\Psi(\rho_m, \rho'_s, \nabla\rho_m, \nabla\rho'_s) \quad (35)$$

which is a working form of eq. (5). The function $\theta(\rho_m, \rho'_s)$ is defined by eq. (21). Eq. (11a) can be used to obtain the mass-frame source term, σ'_s , in eq. (35). Squaring of the mass-frame diffusion kinetics can also be used to construct the dissipation potentials [65].

In the traditional frame, eq. (34), the partial generalized momenta are

$$\mathbf{p}(\rho, \rho_s, \mathbf{u}, \mathbf{u}_s, \eta) = \left(\frac{\partial L}{\partial \mathbf{u}}\right)_{\rho, \rho_s, \mathbf{u}_s} = \rho\mathbf{u} - (\rho_s^2\rho^{-1}g - \eta\rho_s^2\kappa^{-1})(\mathbf{u}_s - \mathbf{u}), \quad (36)$$

$$\mathbf{p}_s(\rho, \rho_s, \mathbf{u}, \mathbf{u}_s, \eta) = \left(\frac{\partial L}{\partial \mathbf{u}_s}\right)_{\rho, \rho_s, \mathbf{u}} = (\rho_s^2\rho^{-1}g - \eta\rho_s^2\kappa^{-1})(\mathbf{u}_s - \mathbf{u}). \quad (37)$$

From Nöther's theorem the total momentum density, $\mathbf{p} + \mathbf{p}_s$, equals to the sum $\mathbf{p} + \mathbf{p}_s$ which yields the mass flow, $\mathbf{p} + \mathbf{p}_s = \rho\mathbf{u}$, as the total momentum density.

In the canonical (mass) frame, eq. (35), the partial momenta are

$$\mathbf{p}_m(\rho_m, \rho_s, \mathbf{u}_m, \mathbf{u}_s, \eta) = \left(\frac{\partial L}{\partial \mathbf{u}_m}\right)_{\rho, \rho_s, \mathbf{u}_s, \eta} = \rho_m\mathbf{u}_m + + (\eta'_s - \phi_m)\theta(\kappa g)^{-1}\rho_m\rho'_s(\mathbf{u}_s - \mathbf{u}_m), \quad (38)$$

$$\mathbf{p}_s(\rho_m, \rho_s, \mathbf{u}_m, \mathbf{u}_s, \eta) = \left(\frac{\partial L}{\partial \mathbf{u}_s}\right)_{\rho, \rho_s, \mathbf{u}_s, \eta} = \rho'_s\mathbf{u}_s + - (\eta'_s - \phi_m)\theta(\kappa g)^{-1}\rho'_s\rho_m(\mathbf{u}_s - \mathbf{u}_m). \quad (39)$$

The governing dissipative effect in the mass frames is friction; note the frictional structure of the dissipative terms. Nöther's theorem implies again that the total density, $\rho_m + \rho_s$, equals the sum of the above derivatives; consequently, $\rho_m + \rho_s = \rho$ again. Clearly, the partial momenta, including the thermal momenta, differ in the two

frames; nonetheless they always yield the same physical quantity, . This momentum density is an invariant of thermo-hydrodynamic transformations applied at a fixed point of the space-time.

A particularly transparent example is the reversible canonical case ($\sigma'_s = 0$) when the phase η'_s is not explicitly present in L and the simplest canonical momenta appear, $\mathbf{p}_m = \rho_m \mathbf{u}_m$ and $\mathbf{p}_s = \rho'_s \mathbf{u}_s$, pertaining to the absolute velocities of the bare mass and thermal mass. In this case each of these momenta is proportional to the corresponding flux and the equality, $\mathbf{p} = \rho \mathbf{u}$ follows in the simplest way possible. According to Sect. 2, these expressions are believed to represent the physical kinetic momenta of thermal mass and bare mass.

The invariance property also holds for the remaining components of the tensor of matter: the energy E , the stress tensor \mathbf{T} , and the energy flux \mathbf{Q} , as well as for any state function of thermo-hydrodynamics. From the common formula

$$E = \sum_l \frac{\partial L}{\partial \mathbf{u}^l} \cdot \mathbf{u}^l - L, \quad (40)$$

the same Nöther's energy function is obtained in both descriptions. In a general irreversible case and with canonical variables used,

$$E = \frac{1}{2} \rho_m \mathbf{u}_m^2 + \frac{1}{2} \rho'_s \mathbf{u}_s^2 + \rho_e(\rho'_s, \rho_m) + (\rho'_s + \rho_m) \psi(\mathbf{x}, t) + (\eta'_s - \phi_m) \theta (\kappa g)^{-1} \rho'_s \rho_m (\mathbf{u}_s - \mathbf{u}_m)^2 - \Psi(\rho_m, \rho'_s, \nabla \rho_m, \nabla \rho'_s). \quad (41)$$

The first line of this equation represents the classical “reversible” energy whereas the terms in the second line together represent the contribution of the relative thermal phase $\eta = (\eta'_s - \phi_m) \theta$. This term contains the negative product of η and the difference between the two Onsagerian dissipation functions, the difference sometimes called the thermodynamic Hamiltonian [77]. In a limiting situation, when the phase η has no effect (or “does not enter”) and the Fourier's law holds ($\Phi = \Psi$), only the first line of eq. (41) plays a role. Such a truncated formula corresponds to the reversible momenta $\rho'_s \mathbf{u}_s$ and $\rho_m \mathbf{u}_m$. In this limit the energy density E is unaffected by the dissipation.

For an arbitrary L the Nöther's theorem yields as the energy flux

$$\mathbf{Q} = \mathbf{u}_m \left(\mathbf{u}_m \cdot \frac{\partial L}{\partial \mathbf{u}_m} - \rho_m \frac{\partial L}{\partial \rho_m} \right) + \mathbf{u}_s \left(\mathbf{u}_s \cdot \frac{\partial L}{\partial \mathbf{u}_s} - \rho'_s \frac{\partial L}{\partial \rho'_s} \right). \quad (42)$$

Hence for L of eq. (5),

$$\begin{aligned} \mathbf{Q} = & \mathbf{u}_s \left[\mathbf{p}_s \cdot \mathbf{u}_s + \left(\frac{T}{\theta} + \mu - \frac{T \rho'_s \theta \rho'_m}{\theta^2} - \frac{\mathbf{u}_s^2}{2} + \eta' \frac{\delta \sigma'_s}{\delta \rho'_s} \right) \rho'_s \right] + \\ & + \mathbf{u}_m \left[\mathbf{p}_m \cdot \mathbf{u}_m + \left(\mu - \frac{T \rho'_s \theta \rho'_s}{\theta^2} - \frac{\mathbf{u}_m^2}{2} + \eta' \frac{\delta \sigma'_s}{\delta \rho_m} \right) \rho_m \right], \end{aligned} \quad (43)$$

with $\eta' = \eta'_s - \phi_m$. As the general formula for \mathbf{Q} is complex, in this case we restrict our compatibility test to the limiting phaseless situation. Then

$$\mathbf{Q} = \mathbf{u}_s \rho'_s \left(\frac{T}{\theta} + \mu - \frac{T \rho'_s \theta \rho'_s}{\theta^2} + \frac{\mathbf{u}_s^2}{2} \right) + \mathbf{u}_m \rho_m \left(\mu - \frac{T \rho'_s \theta \rho_m}{\theta^2} + \frac{\mathbf{u}_m^2}{2} \right). \quad (44)$$

Introducing into the above formula the diffusion velocities of the thermal mass and bare mass, $\mathbf{v}_s = \mathbf{u}_s - \mathbf{u}$ and $\mathbf{v}_m = \mathbf{u}_m - \mathbf{u}$, and the related diffusion fluxes, $\mathbf{j}'_s = \rho'_s \mathbf{v}_s$ and $\mathbf{j}'_m = \rho_m \mathbf{v}_m$ one can distinguish expressions describing the power of the nonequilibrium stresses, \mathbf{w} , and the heat flux, \mathbf{q} . The power \mathbf{w} obeys

$$\mathbf{w} = \Pi \cdot \mathbf{u} + v_s^2 \mathbf{j}'_s + v_m^2 \mathbf{j}'_m = \Pi_s \cdot \mathbf{u}_s + \Pi_m \cdot \mathbf{u}_m \quad (45)$$

corresponding with the total nonequilibrium stress $\Pi = \rho_m \mathbf{v}_m \mathbf{v}_m + \rho'_s \mathbf{v}_s \mathbf{v}_s$ or the sum of the partial stresses accompanying diffusion of the thermal mass and the bare mass, $\Pi_s = \rho'_s \mathbf{v}_s \mathbf{v}_s$ and $\Pi_m = \rho_m \mathbf{v}_m \mathbf{v}_m$. Associated with the power \mathbf{w} is a mass-frame formula for the heat flux

$$\begin{aligned} \mathbf{q} &= \left(-\frac{\delta L}{\delta \rho'_s} - \frac{1}{2} v_s^2 \right) \mathbf{j}'_s + \left(-\frac{\delta L}{\delta \rho_m} - \frac{1}{2} v_m^2 \right) \mathbf{j}'_m \\ &\equiv \left(\frac{T'}{\theta} - \frac{1}{2} v_s^2 \right) \mathbf{j}'_s + \left(\mu' - \frac{1}{2} v_m^2 \right) \mathbf{j}'_m \\ &= \left(\frac{T}{\theta} + \mu - \frac{T \rho'_s \theta \rho'_s}{\theta^2} - \frac{1}{2} v_s^2 \right) \mathbf{j}'_s + \left(\mu - \frac{T \rho'_s \theta \rho_m}{\theta^2} - \frac{1}{2} v_m^2 \right) \mathbf{j}'_m. \end{aligned} \quad (46)$$

Hence

$$\mathbf{q} = [T(1 - \rho \rho_s \rho_m^{-1} \theta \rho'_s) - \frac{\theta}{2} (j_s^2 \rho_s^{-2} - \theta^2 j_s^2 \rho_m^{-2})] \mathbf{j}_s = \tilde{T} \mathbf{j}_s, \quad (47)$$

where \tilde{T} is the kinetic temperature introduced in ref. [58] in the conventional frame. The reduction of the chemical potential in eq. (46), which occurs because $j'_s + j_m = 0$, is crucial for preserving the form $\mathbf{q} = \tilde{T} \mathbf{j}_s$. While the definition (45) is in the mass context more relevant than the classical stress work $\Pi \cdot \mathbf{u}$, the sum of the heat and work remains independent of their definitions. Thus, the final formula for the total energy flux density is the same as that obtained in the conventional frame,

$$\mathbf{Q} = \rho \mathbf{u} \left(\frac{u^2}{2} + h + \frac{\rho_m}{2\rho} v_m^2 + \frac{\rho'_s}{2\rho} v_s^2 \right) + \mathbf{q} + \mathbf{w}. \quad (48)$$

The fact that the same tensor of matter \mathbf{G} is hosted in various frames is important in general relativity where (the relativistic) \mathbf{G} is the unique source of the same gravitational field generated in mass frames and entropy frames.

12. Concluding Remarks

Two general kinds of specification of the thermal mass have been discussed here. One kind is made to satisfy the approaches hypothesizing the constancy of thermal

mass per unit of the entropy, θ . The other general approach supposes the variable θ to be consistent with the Grad-Boltzmann theory. Yet, it is the existence of a significant plateau in the formula for θ based on Grad's solution which allows us to treat the entropy as a direct, approximately linear measure of the thermal mass for a reasonably broad range of state changes.

If intrinsic symmetries and finiteness of the thermal momenta are then required, one must admit creation of thermal mass from the bare mass of particles, in order to preserve global conservation of matter. For the constant- θ hypotheses the creation of the thermal mass is necessarily linked with the equivalent effect of the entropy production. While such an effect is also plausible within Grad's model of variable θ , and even necessary in plateau range of θ , entropy production does not have to be uniquely defined in this case as the source of the thermal mass. The entropy source can nonetheless be determined from the thermal mass source (and conversely) with the formula of eq. (11). Yet, since the standard Onsager-Joule dissipation expressions (too approximate in character) cannot incorporate the effect of the variable θ in a covariant way, the general case of phase dependent dissipative processes seems now to be practically limited to the constant- θ approximation.

The usual thermodynamic variables cease to be the most relevant variables when inertial effects (and associated sources) are present. Even for weak definitions of θ , the thermodynamics of the mass frame is different from (but equivalent to) that in the conventional frame. In the former the temperature becomes an auxiliary variable only, and the basic intensities are the chemical potentials of the thermal mass and the bare mass. With the thermal inertia definitions of the natural variables [41] have to be broadened.

In spite of quantitative disagreement between various definitions, a finite thermal momentum can be interpreted as that of the thermal mass; it is a consequence of the phenomenon that a part of the observed mass of the medium is of purely thermal (i.e. entropy-related) origin. As soon as flow is described in terms of a thermal component and a mass component, the part of the energy flow that is equipartitioned as it moves ahead in the fluid can be associated with a velocity, a momentum and a mass, the thermal velocity, momentum and mass. This division makes the thermal momentum as legitimate a variable as the flow of entropy itself. Any consistent set of the momenta yields the same tensor of matter.

Why introduce and investigate the thermal mass? A different view of the same phenomenon (here dissipation) is always illuminating. Some effects hidden in the classical representation are easier to recognize in the canonical (mass) frames: explicit momenta, representation dependence, nonconservative properties of mass, invariants of classical thermodynamics. All phase variables become velocity potentials in the mass frame; the transformation to this frame reveals that the entropy phase is a relative quantity, which is unaffected by the common phase shift, and

hence has the property of a state variable. The physical nature of η has recently been confirmed in a different approach using a relativistic interpretation of η [79]. One important conclusion of our analysis is that a source in one frame does not need to be the same in another frame if nonlinear transformations are involved. Yet, the state functions, the matter tensor and the total and thermal mass remain unchanged.

Quantum field theories may help to add more precision to the concept of thermal mass. Kandrup [80] argues that the produced entropy possesses an intrinsic physical meaning, this meaning being especially clear in the context of a quantum theory, where a direct connection exists between entropy generation and particle creation. Intraparticle force fields in manybody relativistic systems possess necessarily a finite energy density E^{int} and hence an intraparticle mass E^{int}/c^2 . Identifying E^{int}/c^2 with $\rho'_s = \theta\rho_s$ one can develop a relativistic theory of thermal mass [78].

There are several apparent advantages of using an explicit thermal mass. One argument follows from consistency in operating with all entities represented by the same form as that of the matter, the mass or density form. The exotic notion of the thermal momentum (of the entropy flow) acquires in the canonical frame the natural physical interpretation of the usual linear momentum assigned to mass in motion. The appraisal of the extremum Lagrangian of the fluid Λ as the pressure stemming from the mass constraint of the action integral gains in the mass frame its corroboration. The dissipative aspects of the thermal mass involve the frictional behavior and diffusional models; the latter are known as those exhibiting relatively small variability of their coefficients with the process state. The use of friction coefficients sheds more light on the mechanism of the flux interaction than the use of conductances, which combine diverse friction and concentration effects [81]. The role of a caloric coordinate for consistent setting of an H -theorem in Lagrangian frames is essential, as first exposed by Grmela and Teichmann [82].

However the most important applications of the thermal mass are still to come through its prospective inclusion to the theory of relativistic systems, since the effect of the thermal inertia in terms of entropy has only a formal meaning, whereas that in terms of mass and diagonal kinetic energy allows the direct use of the whole power of relativity theory. In particular, some nagging difficulties of relativistic thermodynamics (e.g. relativistic temperature transformation) can be treated in a new, effective way [78]. Another prospective application is the theory of chemical reactions, where the thermal mass may be regarded as an extra reacting component. In this case, recent treatments imbedding chemical kinetics within the context of nonequilibrium thermodynamics may find original and useful applications in the canonical mass frames.

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