Forbidden Continuum: Free-Bound Transitions in Hydrogen Plasmas

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(Received 1 November 1965)

It is shown that local electric fields in ionized gases can lead to forbidden free-bound transitions. The photodetachment of electrons from $H^+$ is considered, and specifically the cross section for detachment into continuum $s$ waves is calculated from a binary-collision model. The corresponding forbidden radiative attachment is computed. The calculations indicate that in hydrogen plasmas with ion densities of $10^{18}$ cm$^{-3}$ or above, the forbidden continuum should make a large contribution to the near-infrared emission. The effect may also appear in alkali plasmas.

I. OUTLINE OF THE PROBLEM

The photodetachment-radiative capture process of hydrogen atoms and electrons,

$$H^+ + h\nu \rightarrow H^0 + e^-$$  
(1)

has proved to be an important one, not only for its own sake but as a device for studying for the theory and calculation of free-bound transition probabilities$^{1,2}$ as a stimulus for other studies of photodetachment$^4$ and as an important contributor to the radiative properties of the sun$^4$ and other stars$^4$ and of plasmas.$^5-10$ The electron affinity of hydrogen—the threshold energy for the photon of process (1)—is 0.75 eV. Consequently light whose wavelength is 16 400 Å or less can be absorbed by one of the two bound 1s electrons of $H^-$, which then leaves as an outgoing $p$ wave, according to the one-electron selection rule $\Delta l = \pm 1$. The difference between the absorbed energy $h\nu$ and 0.75 eV appears as kinetic energy of the outgoing electron. The continuum states of nonzero angular momentum are rather sparse at low energies. The density of available continuum states is most sharply expressed by the threshold law for the photodetachment cross section $\sigma_d$.$^{11,12}$ If $k$ is the asymptotic momentum of the outgoing electron, in units of $\hbar$, then for low $k$,

$$\sigma_d \propto k^{2l+1}l^{-1},$$  
(2)

where $l_<$ is the lowest angular-momentum quantum number of the allowed states of the outgoing electron. The expression (2), for the leading term in the $k$

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with $l' = l \pm 2$ or more can be excluded from the mixing process. This point is discussed briefly later.)

If a bound orbital is primarily $s$-like but has a little $p$-character, then optically allowed transitions from this orbital occur not only to $p$-orbitals but also to one-electron $s$ and $d$ states. The transition matrix element connecting an initial one-electron state $\psi_1$, with any final state $\psi_{\text{st}}$ (bound) or $\psi_{\text{f}}$ (free) we call \( \langle 1| \mathbf{p} | n \rangle \) or \( \langle 1| \mathbf{p} | k \rangle \). Suppose the initial (perturbed) orbital $\psi_1$ is a mixture of $s$ and $p$ parts: $\psi_1 = \alpha \psi_s + \beta \psi_p$, $\alpha \gg \beta$; then the cross section for a transition from $\psi_1$ to an $s$ wave is proportional to $\beta^2$:

$$
\sigma_{s \rightarrow st}(\nu) \propto \left| \langle 1| \mathbf{p} | k \rangle \right|^2 \rho_{st}(\nu)
$$

or

$$
\beta \left| \int \overline{\psi_p} \psi_{st} \right|^2 \rho_{st}(\nu).
$$

We have explicitly included $\rho_{st}(\nu)$, the density of one-electron states with angular momentum $l$ and with momentum $k$, corresponding to a photon frequency $\nu$.

Near the threshold $\nu_{p0} \rho_{p0}$ is small, so that even though $\alpha$ is large, $\sigma_{s \rightarrow p0}$ is small for process (1). On the other hand $\rho_{st}$ is relatively large. The question we have now examined is this: Under realistic conditions, do $\beta$ and $\int \overline{\psi_p} \psi_{st}$ get large enough to make $\sigma_{s \rightarrow p0}$ comparable to $\sigma_{s \rightarrow p}$? Could one expect to observe $s$-wave threshold behavior for photodetachment or radiative capture in the spectrum of a hydrogen plasma or a star?

Forbidden transitions of just the kind we have been describing have been described for many years in line spectra. Bound-bound transitions between $s$ and $d$ states and between two $s$ states were observed in potassium and shown to be field induced forbidden transitions.

The absorption spectra of shock-heated salt vapors exhibited some alkali $s$-$s$ and $s$-$d$ transitions also. There is little problem observing such forbidden lines because they frequently are well apart from other lines. Recognizing a forbidden continuum would seem to be a different matter, at least at first glance. The one property by which one might hope to identify a forbidden continuum would be a very significant and characteristic contribution to the contour of the continuous spectrum—for example a drastic change in the threshold shape. Our question must be amplified: Can we expect to identify any forbidden component of the continuous spectrum associated with process (1)?

The conclusion we reach is yes; the hydrogen photodetachment or radiative capture continuum can be expected to contain a significant and unambiguously recognizable contribution from forbidden $s$ to $s$ transitions. This forbidden component is particularly important, naturally, in the near-infrared region. Its contribution is large enough and characteristic enough that there should be little difficulty in observing and identifying the forbidden continuum from its energy dependence near threshold. The remainder of this article describes the means we used for computing the cross section for the process, the limitations and shortcomings of these particular means, and the results of the calculations, the value of the cross section as a function of the photon energy, the temperature of the system and the ion density.

II. CALCULATION OF THE CROSS SECTION

A. The Model

The calculation was carried out in terms of the photodetachment cross section, i.e., process (1) from left to right. In this way it could be carried out as a two-body problem, involving only the negative hydride ion and a single perturbing ion immersed in the radiation field. The cross section $\sigma_{\text{att}}$ or attachment coefficient $\sigma_{\text{att}}$ for the reverse process of radiative capture is related to that for photodetachment by the condition of microscopic reversibility and $\sigma_{\text{att}}$ was so computed.

The detachment cross section is

$$
\sigma_2(\nu) = \frac{\alpha}{\pi} \frac{1}{\nu} \left| \langle 1| \mathbf{p} | k \rangle \right|^2,
$$

where $\mathbf{p}$ is the full electric dipole operator $\sum r_i e r_i$ or $\sum p_i$. We use the dipole length form for the cross section, rather than the momentum (dipole velocity) form. Moreover we have used small capitals and Roman numerals to indicate the full two-electron wave functions, and reserve ordinary lower case type for one-electron functions.

The cross section is an ensemble average, taken over all the perturbations to which each hydride ion is subject. For our purposes, we can use a binary-collision model, treat each collision in terms of its classical adiabatic Coulomb trajectory and then integrate the cross section per trajectory over all trajectories—i.e., over all impact parameters and velocities. An alternative approach would use a static calculation based on the Holtsmark field distribution or a similar field distribution modified to account for correlations in the ion distribution functions. However for digital-computer calculation, the trajectory method was at least as simple.

In a single collision, the instantaneous transition probability is a function only of $k$ (or $l$) and of $R$, the distance between the absorbing hydride ion and the perturber. The contribution to the cross section due to a single collision is obtained by integration of the instantaneous cross section along the trajectory. The
The functional form of the instantaneous transition probability depends on the choice of both bound and continuum wave functions.

### B. The Wave Functions

The bound-state wave function we chose is the variational function used by Wikner and Das to compute the polarizability of H$^-$.[2] The two-electron function $|\psi\rangle$ is an antisymmetrized product of two polarized orbital functions $\psi$:

$$ |\psi\rangle = |1\rangle = N_1 \alpha \psi(1) \psi(2), $$

(bar implying $\beta$ spin and no bar, $\alpha$ spin),

$$ \psi(1) = \psi_{s1}(1)[1 - H_1(1)(\alpha + br_1)] $$
$$ = \psi_{s1}(1) + \psi_{p1}(1), \quad \text{the perturbed orbital}, $$

$$ \psi_{s1}(1) = (N_2/\sqrt{2})[e^{-rs1} + e^{-rs1}], $$

the unperturbed orbital, and

$$ H_1 = -2r_1 \cos \theta / R^2, $$

$$ \frac{H_1}{2} \quad \text{the instantaneous perturbation Hamiltonian}. $$

The constants $N_2$, $\alpha$, $\beta$, $q$, and $c$ are the following:

$$ N_2 = 0.454, $$

$$ a = 0.819, $$

$$ b = 0.672, $$

$$ q = 2.63, $$

$$ z = 0.462, $$

and

$$ c = 2.04. $$

The operator $\alpha$ is an antisymmetrizer and $N_2$ is the normalization constant for $\psi_{s1}$.

(There is actually no real need to antisymmetrize because we are concerned only with one-electron operators and orthogonal functions. At no point have we included exchange effects in the determination of the wave function.)

The interparticle distance dependence appears in the renormalizing factor,

$$ N_1 = R^4 / (R^4 + 7.0 \times 10^6) $$

in atomic units of length, 0.529 Å.

The final-state wave function is again a product, this time of a ground-state hydrogen atom function and a continuum one-electron function. The continuum function is an $s$ wave, a spherical Bessel function orthogonalized to the initial unperturbed one-electron hydride orbital. In the two-electron function, we denote only the excited electron explicitly. Thus, the full final-state function is

$$ |KS\rangle = |KS_0\rangle - |t_0\rangle |t_0\rangle |KS_0\rangle $$

and

$$ |KS_0\rangle = N_s \alpha (\sin kr_s / r_s) X(r_s) \left[ \alpha(1) \beta(2) - \alpha(2) \beta(1) \right], $$

where

$$ X(r) = e^{-r} $$

and in energy normalization,

$$ N_s = 1 / (k_s)^{1/2}. $$

The overlap integral

$$ \langle t_0 | KS_0 \rangle = 16 \pi^2 N_s^4 \left[ \frac{3!}{(1 + q_s^4)^3} \right] $$

$$ \times \left[ \frac{c3!}{(1 + q_s^4)^3} \right], $$

and for $k \ll 1$,

$$ \langle t_0 | KS_0 \rangle = 16 \pi^2 N_s^4 \times 10.92 k_s^{-3} (1 + q_s^{-2}). $$

The one-electron spin functions are $\alpha(i)$ and $\beta(j)$.

Figures 1 and 2 show parts of the wave functions—the unperturbed hydride orbital, the $p$ part of the perturbed hydride orbital, and the spherical Bessel function of the continuum. Note especially how the bound-state functions and especially the $p$ part extend to larger $r$. We

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*E. G. Wikner and T. P. Das, Phys. Rev. 107, 497 (1957).*
shall return to this when we discuss the range of validity of the threshold law.

C. The Cross Section

The total dipole matrix element \( \langle t|z_1+z_2|ks \rangle \) is conveniently broken into two terms, corresponding to the simple spherical wave part, \(|ks_0\rangle\), and to the orthogonalizing part \(|i_1/10|ks_0\rangle\), as follows:

\[
\langle t|z_1+z_2|ks \rangle = 2(T_1 + T_2),
\]

where

\[
T_1 = \frac{16\pi^3}{3} \frac{N_k N_s N_s}{R^3} \left[ \frac{3!}{(1+z)^4} + \frac{c^3!}{(1+q^2)^4} \right] \frac{\alpha^4!}{(s^2+k^2)^{1/2}} \sin\left( 5 \tan^{-1} \frac{k}{z} \right) + \frac{\alpha^4!}{(q^2+k^2)^{1/2}} \sin\left( 5 \tan^{-1} \frac{k}{q} \right)
+ \frac{\beta^5!}{(s^2+k^2)^2} \sin\left( 6 \tan^{-1} \frac{k}{z} \right) + \frac{\beta^5!}{(q^2+k^2)^2} \sin\left( 6 \tan^{-1} \frac{k}{q} \right)
\times \left[ \frac{2k(3s^2-k^2)}{(s^2+k^2)^2} + \frac{2k(3q^2-k^2)}{(q^2+k^2)^2} \right]
\]

Numerically,

\[
T_1 = \left[ R^3/(R^4+7 \times 10^4) \right] f_1(k)
\]

and

\[
f_1(k) = 6.27k^{-1/2} \left[ \frac{35.9}{(0.213+k^2)^{1/2}} \sin(5 \tan^{-1}2.16k) + \frac{73.2}{(1.49+k^2)^{1/2}} \sin(5 \tan^{-1}0.82k) + \frac{147}{(0.213+k^2)^{1/2}} \sin(6 \tan^{-1}2.16k)
+ \frac{300}{(1.49+k^2)^{1/2}} \sin(6 \tan^{-1}0.82k) + 49.9 \left( \frac{2k(0.619-k^2)}{(0.213+k^2)^2} + \frac{4.08k(4.47-k^2)}{(1.49+k^2)^2} \right) \right].
\]

The second term \( T_2 \), based on the orthogonality of \(|i_1\rangle\) and \(|ks\rangle\), is

\[-\langle t|z_1|10s|10s|ks_0\rangle,
\]

which is nonvanishing because \(|10s\rangle\) is polarized by the passing ion and has a transient nonzero dipole moment. The first factor of \( T_2 \) which is independent of \( k \), reduces to

\[
2\pi N_k N_s N_s \left[ \frac{6\alpha}{(2s)^7} + \frac{\epsilon^6\alpha}{(2q)^7} + \frac{2\epsilon^6\alpha}{(s+q)^7}
+ \frac{7\beta}{(2s)^4} + \frac{\epsilon^7\beta}{(2q)^4} + \frac{2\epsilon^7\beta}{(s+q)^4} \right].
\]

The second factor, which was obtained previously, is given by

\[
16\pi^3 N_k N_s N_s \left[ \frac{c^3!}{(1+z)^4} + \frac{3!}{(1+q^2)^4} \right]
\times \left[ \frac{2k(3s^2-k^2)}{(s^2+k^2)^2} + \frac{c^2k(3q^2-k^2)}{(q^2+k^2)^2} \right].
\]

Now upon numerical evaluation the product of (16) and (17) becomes

\[-T_2 = \left[ R^3/(R^4+7 \times 10^4) \right] f_2(k),
\]

where

\[
f_2(k) = 1.14 \times 10^3 \left[ \frac{2k(0.619-k^2)}{(0.213+k^2)^2}
+ \frac{4.08k(4.47-k^2)}{(1.49+k^2)^2} \right].
\]

Typically, if the photon wavelength is 16 000 Å, the one-electron transition dipole gives us

\[ T_1 = 2.49 \times 10^6 \frac{R^3}{(R^4+7 \times 10^4)}. \]

The final expression for the forbidden \( 1s \rightarrow ks \) cross section is

\[ \sigma(R) = \frac{[R^4/(R^4+7 \times 10^4)]^3 f_1(k) f_2(k),} \]

where \( f_1(k) \) is given by Eq. (15) and \( f_2(k) \) by (19).

The collisions with initial relative velocity within \( \delta v \), which and with impact parameters within \( \delta b \) about \( b \) contribute a sort of differential to the total cross section at frequency \( \nu \) (or final momentum \( k \))

\[ \sigma(v,b,v)dvdb = P(v)dvP(b)dbv \int_{-\infty}^{\infty} \sigma(R(v))dv, \]

where \( P(v) \) and \( P(b) \) are, respectively, the distribution functions for the velocity and impact parameters:

\[ P(v) = 4\pi v^2 (m/2\pi kT)^{3/2} e^{-mv^2/2kT}, \]

\[ P(b) = 2\pi b, \]
and \( P \) is the number density of perturbing ions. The time integration is taken along the classical trajectory determined by \( v \) and \( b \).

The total cross section for a given outgoing momentum \( k \) or energy \( h^2k^2/2m \) is

\[
\sigma(k) = \int_0^\infty dv P(v) \int_{b_0}^{b_1} db 2\pi b \sigma(R(t)) dt. \tag{24}
\]

We have made the time dependence of \( R \) explicit in Eqs. (21) and (24). The function \( \sigma(R) \) can be expressed as

\[
\sigma(R) = A(\lambda) R^4/(R^4 + 7 \times 10^3)^3. \tag{25}
\]

Values of \( A(\lambda) \) are given in Table I.

**Table I.** Coefficients \( A \) for cross sections \( \sigma(R) \), for various wavelengths, and for fixed H-perturber distances \( R \): \( \sigma(R) = A(\lambda) R^4/(R^4 + 7 \times 10^3)^3 \).

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>16 000</th>
<th>15 500</th>
<th>13 000</th>
<th>10 000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A(\lambda) )</td>
<td>4.8\times10^9</td>
<td>7.7\times10^8</td>
<td>2.1\times10^9</td>
<td>4.0\times10^9</td>
</tr>
</tbody>
</table>

The innermost integral, the time integral, was calculated by evaluating the integral of \( \sigma[R(t)] \) at 300 points along the Coulomb trajectory, and then integrating numerically. The integral becomes

\[
\sum_i t_i \sigma[R(t_i)],
\]

where

\[
t_i = \Delta t/v_i(t),
\]

\( v_i = \) average velocity in the interval \( t_i \),

and

\[
\Delta s = [(\Delta x)^2 + (\Delta y)^2]^{1/2} \text{ along the trajectory defined by } m \nu_0^2 s - \nu_0^2 b^2 = 1.
\]

The instantaneous velocity \( v \) is related to the initial velocity \( v_0 \) by the energy-conservation condition

\[
v = (v_0^2 + 2/mR)^{1/2}.
\]

For our degree of approximation, we only need consider the collisions between oppositely charged particles.

The error we make by neglecting the collisions of negative ions with each other is less than a factor of 2 because of ion correlations, and we can only expect our calculated transition probability to be within about a factor of 2 or 3 of the true value.

The actual limits on the trajectory integrations were taken at the point of closest approach or 20 atomic units (a.u.), whichever was larger, and out to roughly 2000 a.u., for most paths for the outer limit.

The integrations over impact parameters were computed numerically for 39 impact parameters. The smallest value of \( b \) was \( b_0 \) which was fixed at 20 a.u. This was required because our method is not suitable for treating close collisions. The largest impact parameter was 1900 a.u., comparable to the Debye radius of the system at a density of about \( 10^{13} \) particles/cm\(^2\) and a temperature of about 4300\(^\circ\)K.

The velocity integration was done graphically. A plot of a typical integrand for the velocity integration is shown in Fig. 3. The most convenient form for this step is the reduced variable obtained by removing both the wavelength and total-ion-density dependence.

Finally, then, we obtain the \( 1s-k\_s \) photodetachment cross section in our approximate picture, based on (1) binary collisions, (2) classical trajectories, (3) positive ion perturbers, (4) adiabatic perturbations, (5) a transition probability calculated with approximate wave functions in dipole-length formalism, (6) neglect of continuum \( d \) states, and (7) omission of effects of close collisions. We make one further assumption to calculate the total cross section, namely that this total can be estimated as just the sum of parts from \( s \) and \( p \) continuum functions. In other words we neglect the interference term due to perturbation of the continuum by passing ions.\(^{29}\) Hence

\[
\sigma_d(v) = \sigma_d(1s \rightarrow kp,v) + \sigma_d(1s \rightarrow ks,v).
\]

The forbidden cross section was calculated for wavelengths from 16 000 to 5000 \( \AA \). The part coming directly from the spherical Bessel function is considerably smaller than the part due to the orthogonality imposed on the continuum function. The former or \( f_1 \) term reaches its maximum at about 15 500 \( \AA \), and the \( f_2 \) term, at about 5500 \( \AA \). (The allowed-term maximum is at about 8250 \( \AA \).) The position of the maximum is likely to be a moderately sensitive function of the wave functions used for the calculation, so the calculated value should only be taken as a qualitative indication that the forbidden contribution may be important over the same range as the allowed contribution, whenever it is important at all.

\(^{29}\) This is a difficult term to estimate, because the final-state wave function is rather complicated. At small \( r \), the one-electron continuum function is approximately a solution of the Schrödinger equation for an \( r^{-1} \) potential centered about the now-neutral hydrogen. Then, at larger \( r \), it must become a Coulomb continuum function centered about the perturbing ion. One can hardly use perturbation theory to estimate such a function, especially at small distances.
The forbidden, allowed and total cross sections for the photodetachment process (1) are shown in Fig. 4, for one set of conditions that can be obtained relatively easily in laboratory plasmas, $2.5 \times 10^{14}$ ions/cm$^2$ at a temperature of $3000^\circ$K. Note that the forbidden contribution to $\sigma$ increases linearly with ion density. Some values are also given in Table II.

The temperature dependence of the forbidden cross section is naturally small but should be detectable. This dependence is shown in Fig. 5, for two values of $k$.

The threshold behavior of the forbidden cross section is that appropriate to an outgoing $s$ wave. Up to about 11 000 Å, the cross section is linear in $k/\lambda$, and begins to deviate from the threshold law there. Figure 6 shows the dependence of the forbidden cross section on $k/\lambda$. If the continuum function had not been orthogonalized to the bound-state function, the linear part would be much shorter. In fact the unorthogonalized function gives a $\sigma$ that deviates from the threshold law a bit even at 16 000 Å. This is a simple consequence of the very high polarizability of $H^-$. The perturbed part of the ground-state wave function of a polarized $H^-$ extends to very large distances, as Fig. 2 shows. The transition dipole gets contributions from these large distances, 16 a.u., and beyond. Even for very low $k$, say 0.05, $\sin kr$ is not well approximated by $kr$. This means that for $k > 0.05$, the dipole term connecting $\sin kr/r$ with the bound $p$ function is not described accurately by the first term in the threshold expansion. The orthogonalizing term, by contrast, depends on the overlap of $\sin (kr/r)$ with the bound $p$ function, and not on the dipole operator. This term does not weight so heavily the outer parts of the product of $\sin(kr/r)$ and the bound $p$ function. Instead, it gets its large-$r$ contributions in the weakly $k$-dependent transition term connecting the bound $s$ and $p$ functions. As a result, the orthogonalizing term obeys the threshold law to reasonably large values of $k$.

**Table II.** Photodetachment cross sections* for process (1). (Perturber density $2.5 \times 10^{14}$ cm$^{-4}$.)

<table>
<thead>
<tr>
<th>$T = 3000^\circ$K</th>
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<tbody>
<tr>
<td></td>
<td>16 000</td>
<td>15 500</td>
<td>13 000</td>
<td>10 000</td>
<td>9000</td>
<td></td>
</tr>
<tr>
<td>forbidden</td>
<td>$0.10 \times 10^{-17}$</td>
<td>$0.17 \times 10^{-17}$</td>
<td>$0.45 \times 10^{-17}$</td>
<td>$0.85 \times 10^{-17}$</td>
<td>$0.98 \times 10^{-17}$</td>
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</tr>
<tr>
<td>allowed</td>
<td>$0.10 \times 10^{-17}$</td>
<td>$0.30 \times 10^{-17}$</td>
<td>$1.75 \times 10^{-17}$</td>
<td>$3.74 \times 10^{-17}$</td>
<td>$4.03 \times 10^{-17}$</td>
<td></td>
</tr>
<tr>
<td>total</td>
<td>$0.20 \times 10^{-17}$</td>
<td>$0.47 \times 10^{-17}$</td>
<td>$2.20 \times 10^{-17}$</td>
<td>$4.59 \times 10^{-17}$</td>
<td>$5.01 \times 10^{-17}$</td>
<td></td>
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<tbody>
<tr>
<td></td>
<td>16 000</td>
<td>15 500</td>
<td>13 000</td>
<td>10 000</td>
<td>9000</td>
<td></td>
</tr>
<tr>
<td>forbidden</td>
<td>$0.068 \times 10^{-17}$</td>
<td>$0.11 \times 10^{-17}$</td>
<td>$0.30 \times 10^{-17}$</td>
<td>$0.58 \times 10^{-17}$</td>
<td>$0.67 \times 10^{-17}$</td>
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<tr>
<td>allowed</td>
<td>$0.10 \times 10^{-17}$</td>
<td>$0.30 \times 10^{-17}$</td>
<td>$1.75 \times 10^{-17}$</td>
<td>$3.74 \times 10^{-17}$</td>
<td>$4.03 \times 10^{-17}$</td>
<td></td>
</tr>
<tr>
<td>total</td>
<td>$0.17 \times 10^{-17}$</td>
<td>$0.41 \times 10^{-17}$</td>
<td>$2.05 \times 10^{-17}$</td>
<td>$4.32 \times 10^{-17}$</td>
<td>$4.70 \times 10^{-17}$</td>
<td></td>
</tr>
</tbody>
</table>

* Units of cross sections are cm$^2$.
The final figure, Fig. 7, shows the coefficient for the radiative attachment process, (1) read from right to left, for the $4650^\circ$K and $2.5 \times 10^{14}$ ions/cm$^3$. The forbidden, allowed, and total attachment coefficients are all included.

It appears that the contribution of forbidden continuum radiation is probably quite significant in the near-infrared emission of hydrogen plasmas. The phenomenon is presumably recognizable by the increased total emission above the allowed part, and perhaps easier to recognize by the characteristic shape that the forbidden term gives to the threshold. The phenomenon was calculated here for hydrogen, but should be important as well for any other negative ion whose “extra” electron is in an $s$ orbital. For example one might hope to see the same effect in the alkalis. Lithium, sodium or the other alkalis could very easily show the same behavior in their plasmas at moderate densities.

III. DISCUSSION OF APPROXIMATIONS

A. Binary Collisions

The main contribution to the forbidden continuum comes from the close approach portions of the trajectories. We arbitrarily say that the effective collision radius $R_e$ is about 500 a.u., for the purpose of estimating the ratio of two-body to three-body collisions. Then $n^3 = R_e^3$ when $n \approx 6 \times 10^{18}$, so that we can expect our binary collision model to be a reasonably good one for ion densities well above $10^{18}$ cm$^{-3}$.

B. Adiabatic Classical Trajectories

The adiabatic classical path approximation is valid if $|\alpha/h| \gg 1$, and if $|m
u r_0/h| \gg 1$, where $r_0$ is a characteristic minimum size. The velocities considered in the present calculation fall between $10^{-3}$ and $10^{-2}$ in units of $h/\alpha$, or between $4 \times 10^6$ and $4 \times 10^8$ cm/sec. The first condition, for adiabatic collisions, is fulfilled easily; the second, that the de Broglie wavelength be short, is fulfilled at all but the very lowest velocity for protons, and for all velocities for any other positive ions.

C. Dipole-Length Approximation

This is probably the most severe approximation after the use of approximate wave functions. It may introduce an error of as much as a factor of 2 or 3 in the cross section. The accuracy of this approximation depends sensitively on the accuracy of the wave functions, because the change from momentum to position operators in the transition matrix element depends on the assumption that the wave functions are exact. Moreover the dipole-length form gives predominant weight to the outer part of the wave functions, where they are not exceptionally accurate.

D. Close Collisions

The cross sections calculated are more likely too low than too high because of the manner in which the integration over trajectories was done. Those trajectories which had a minimum distance of less than 20 a.u., from the hydrogen ion were truncated at 20 a.u.; that is to say $\sigma(k)$ was calculated along a path until the perturber reached 20 a.u., from the $H^+$. At this point the perturber “hit a hard shell” and was turned back. We know that within this core there will be a definite contribution to the cross section. Since, in the inner region, all the simplifying features of the calculation break down (and we have already indicated the dominance of the forbidden cross section over the allowed under appropriate circumstances), it seemed better to omit the inner region altogether than to treat it in a very inadequate way.

E. Contribution from $d$ Waves

The outgoing wave function should contain some $d$ character as well as $s$ and $p$; however, we have assumed that we can ignore the contribution to the cross section from the “$p \rightarrow d^*$” matrix element. We have seen that the threshold law predicts that for low $k$, $\sigma \approx \nu k^{4\gamma}$. Therefore the low $k$ matrix elements of such a type will go as $k^4$ with $k \ll 1$ while “$p \rightarrow s^*$” matrix elements go as $k$. Hence we can ignore $d$ waves for they will be insignificant at low $k$, so long as we are concerned with the threshold region and only with a very approximate cross section there.

ACKNOWLEDGMENT

This work was supported in part by the U. S. Army Research Office, Durham.
where $\alpha = \pm 1$, depending on whether the ring normal is aiming to the left or the right of $r$. The curvature of the line has been neglected in writing Eqs. (B6). Inserting the expressions in Eqs. (B6) into Eqs. (B4) and (B5), and expanding in powers of $s$, we obtain, after some complicated arithmetic,

$$v_2(r_1,0) = \left(\frac{\kappa R}{2\pi}\right)\{(\alpha h_y, -\alpha h_z)ZB_1 + (h_x, h_y)(RB_0 - PB_1)\} \quad \text{(B7)}$$

and

$$\frac{\partial^2 v_1(r_1,s)}{\partial t^2} \bigg|_{s=0} = \frac{\kappa R}{2\pi} \left\{ (\alpha h_y, -\alpha h_z) \left( \frac{Z}{p} - \frac{Z - C_1}{p^2} \right) + (h_x, h_y) \left( \frac{R}{p} - 3C_0 - \frac{B_1}{p} \right) \right\} \quad \text{(B8)}$$

where

$$C_s = \frac{\int_0^\pi \left( -P + R \cos \theta \right) \cos n \theta d\theta}{\left( Z^2 + P^2 + R^2 - 2PR \cos \theta \right)^{3/2}}. \quad \text{(B9)}$$

In Eqs. (B7), (B8), and (B9), $P$ is meant to be evaluated at $s=0$.

To check the effect of taking some of the neglected terms into consideration, computations were also carried out using $R + \alpha_1 \cos \theta$ instead of $R$ in Eq. (B3). It was found that this way of taking the first distortion term into account produced a completely negligible correction. A more accurate form of Eqs. (B6) which took account of the line vector curvature near the $x-y$ plane was also tried, and also produced only a small correction. This is consistent with the fact that the radius of curvature of the line was always found to be much greater than the ring-line distance.

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**Erratum**

**Forbidden Continuum: Free-Bound Transition in Hydrogen Plasmas, Michael C. Weinberg and R. Stephen Berry [Phys. Rev. 144, 75 (1966)].** A factor of $2\pi$ was missing in the expressions for $\sigma(R)$ or for $A(\lambda)$. As a result, the entries in Table I should be $30.2 \times 10^9$, $48.4 \times 10^9$, $13.2 \times 10^9$, and $25.1 \times 10^9$. Furthermore, the values in Table II, Figs. 6, and 7 refer to perturber densities of $4 \times 10^{13}$ ions/cm$^3$; in Fig. 4, to $2 \times 10^{14}$ ions/cm$^3$, and in Fig. 5, to $4 \times 10^{14}$ ions/cm$^3$. The corrected values thus show a larger effect due to forbidden continuum than did those originally published.