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Analytical and numerical estimates of efficiency for an irreversible heat engine with distributed working fluid

Vladimir N. Orlov and R. Stephen Berry
Department of Chemistry and The James Franck Institute, The University of Chicago, 5735 South Ellis Avenue, Chicago, Illinois 60637

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Previous work on estimating power output from heat engines is extended to estimating efficiency under finite-time constraints. A numerical method for estimating efficiencies for a class of finite-time thermodynamic models with variable-temperature reservoirs and arbitrary heat-exchange laws is proposed. The method is based on energy and entropy balances, and on the use of optimal control, Lagrange multipliers, and convex optimization. It is applicable both to lumped-parameter and distributed-parameter models. An analytical expression for the upper bound of the efficiency is obtained for constant-temperature reservoirs and Newtonian heat-exchange laws. Results of computations for two efficiency definitions are presented.

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I. INTRODUCTION

This article is a computationally oriented extension of the work in Ref. [1]. Here we use the same models, technique, and notations. In Ref. [1] general analytical expressions for estimates of maximum power of a "Carnot-like" heat engine were presented. These expressions were obtained for the Newtonian heat-exchange law by solving auxiliary averaged optimal control problems with one constraint arising from the periodicity of the system's entropy. Here we use analogous but more complicated averaged optimal control problems for the complementary problem of estimating maximum efficiency for a general heat-transfer law. These problems have two constraints. The first constraint is again connected with the periodicity of entropy of the working fluid; the second constraint is either a fixed average power of the process or a fixed average heat flow into the working fluid. For temperature-varying reservoirs and heat-exchange laws that need not be Newtonian, these two-constraint problems are hard to solve analytically, but can be made easy to solve numerically, because, as we show here, they can be reduced to two-dimensional convex optimizations.

For constant-temperature reservoirs and Newtonian heat-transfer laws we have obtained analytical expressions for estimates of maximum efficiency. This is done with the help of a general analytical expression for a maximum-power estimate from Ref. [1]. Analytical expressions for estimates of efficiency are valid for both a distributed-parameter model (in which the working fluid is described by partial differential equations) and a lumped-parameter (uniform-T) model. The analytical expressions are used here to check computations.

Computations carried out for the lumped-parameter model show the growth of efficiency and power output with the growth of the amplitude of the temperature variation of the reservoirs. We use two common definitions of efficiency and compare the results of efficiency bounds computations for them. These definitions imply the same efficiency bounds for constant-temperature reservoirs. Computations show that, for time-varying temperatures of the reservoirs, one may get different results from different definitions of efficiency.

II. FORMULATION OF EFFICIENCY ESTIMATION PROBLEMS

We summarize the formulation briefly here. Details and explanations can be found in Ref. [1].

The average power output of the heat engine we analyze, described by the distributed-parameter model, has the form

$$\mathcal{P}=\frac{1}{\tau}\int_0^{\tau}\left[v_H\int_{A_H} q_H(\xi,T,T_H)da + v_L\int_{A_L} q_L(\xi,T,T_L)da\right]dt\ , \qquad (1)$$

where $q_H(\xi,T,T_H)$ and $q_L(\xi,T,T_L)$ are the heat-exchange laws between the working fluid and the reservoirs, $T_H(t,\xi)>T_L(t,\xi)$ are the temperatures of the engine's two reservoirs. $T(t,\xi)$ is the temperature of the working fluid, and $v_H(t)$ and $v_L(t)$ are the switching functions. These switching functions regulate the finite-rate heat transfer between the working fluid and the two heat reservoirs. When $0<v_H(t)\leq 1$ the high-temperature reservoir and the working fluid are in contact and exchange energy; when $v_H(t)=0$ there is no exchange between them. When $0\leq v_L(t)\leq 1$ the low-temperature reservoir and the working fluid are in contact and exchange energy; when $v_L(t)=0$ there is no exchange between them. Functions v_H, v_L and the time-varying shape of the boundary are considered control variables. We assume that $q_H>0$ when $T<T_H$, $q_H<0$ when $T>T_H$, $q_H=0$ when $T=T_H$, and that $q_L>0$ when

$T < T_L$, $q_L < 0$ when $T > T_L$, $q_L = 0$ when $T = T_L$. For Newtonian (linear) heat-exchange laws $q_H = \alpha_H(\xi)(T_H - T)$, $q_L = \alpha_L(\xi)(T_L - T)$, where $\alpha_H(\xi)$, $\alpha_L(\xi)$ are the space-dependent heat-transfer coefficients. Equation (1) is a corollary of the partial differential equations describing the working fluid, the boundary conditions, and the condition $E(0) = E(\tau)$. Here $E(\tau)$ is the total energy of the working fluid at time t; the period τ is given; $A_H(t)$, $A_L(t)$ are the contact surfaces with area element da, and integration in Eq. (1) is carried out over the area of the surfaces A_H, A_L.

The net amount of heat the working fluid receives from the high-temperature reservoir is

$$Q_H = \int_0^\tau v_H \int_{A_H} q_H da\, dt \ . \qquad (2)$$

The corresponding efficiency, net work per unit of heat from the hot reservoir, is

$$\eta_1 = \tau\mathcal{P}/Q_H \ . \qquad (3)$$

It is possible to define the amount of heat the working fluid receives from the high-temperature reservoir another way, as just the influx, without subtracting any return to T_H arising from any conditions in which T becomes greater than T_H:

$$Q_H^+ = \int_0^\tau v_H \int_{A_H} \tfrac{1}{2}(q_H + |q_H|) da\, dt \ . \qquad (4)$$

An analogous definition of heat received was first used in Ref. [2]. Note that $Q_H^+ \geq Q_H$. Situations when the local temperature of the working fluid is greater than T_H may occur, for example, when shock waves travel through the working fluid in a nonuniform system, requiring a distributed-parameter model. Another situation could arise if the working fluid is heated by the reservoir T_H, further compressed adiabatically and then brought into contact with the hot reservoir again.

Definition (4) implies a second definition of efficiency:

$$\eta_2 = \tau\mathcal{P}/Q_H^+ \ . \qquad (5)$$

It is easy to see that for $Q_H > 0$ we have $\eta_2 \leq \eta_1$.

Control variables which can be used to make the total energy and total entropy periodic are called admissible. We cannot expect to find an analytic expression for the admissible controls that yield optimum efficiency in the distributed-parameter model because the problem is too complex and too general. Instead, we shall consider here three simpler problems:

(1) To find an upper bound of η_1 with constraints $\tau\mathcal{P} = \tau\mathcal{P}^0$, $S(0) = S(\tau)$, i.e., with fixed work per cycle.

(2) To find an upper bound of η_1 with constraints $Q_H = Q_H^0$, $S(0) = S(\tau)$, i.e., with fixed net heat input per cycle.

(3) To find an upper bound of η_2 with constraints $\tau\mathcal{P} = \tau\mathcal{P}^0$, $S(0) = S(\tau)$, i.e., with fixed work per cycle.

Here $S(\tau)$ is the total entropy of the working fluid and $\mathcal{P}^0$, Q_H^0 are given positive constants (the average power output and the amount of heat received correspondingly). We would like to find physically meaningful upper bounds of efficiency, so these bounds should be lower than the Carnot efficiency and should depend upon the admissible control. These problems are equivalent to the following problems:

(1′) To find an upper bound of $-Q_H$ with constraints $\tau\mathcal{P} = \tau\mathcal{P}^0$, $S(0) = S(\tau)$.

(2′) To find an upper bound of $\tau\mathcal{P}$ with constraints $Q_H = Q_H^0$, $S(0) = S(\tau)$.

(3′) To find an upper bound of $-Q_H^+$ with constraints $\tau\mathcal{P} = \tau\mathcal{P}^0$, $S(0) = S(\tau)$.

We allow simultaneous contacts of the working fluid with both the reservoirs, so the admissible controls $v_H(t)$ and $v_L(t)$ meet conditions $0 \leq v_H \leq 1$, $0 \leq v_L \leq 1$.

The cycle-closing condition $S(0) = S(\tau)$ is equivalent to

$$\int_0^\tau \left[v_H \int_{A_H} \frac{q_H}{T} da + v_L \int_{A_L} \frac{q_L}{T} da + \sigma(t) \right] dt = 0 \ . \qquad (6)$$

where $\sigma(t) \geq 0$ is the integrated over the volume entropy production in the working fluid [1].

Our procedure is not to solve the formulated problems directly, but to transform them with the help of entropy balance and constraint breaking to averaged problems. The last problems are transformed with the help of Lagrange multipliers into convex optimizations.

Remark 1. The formulation of the problems will not be changed if the engine has an additional inner heat reservoir (such as the thermal sponge of the Stirling engine). In this situation we could consider Eqs. (1) and (6) as the energy and the entropy balances for the system composed from the working fluid and the inner reservoir. A non-negative term will be added to the entropy production, which is not important to our analysis because we shall use here the only fact that the entropy production is non-negative. When the temperature of this inner heat reservoir is known, one could easily add corresponding terms to Eqs. (1) and (6) and get more precise estimates for the power and the efficiency using direct analogy with the result presented in Ref. [1] and the algorithms below.

III. TRANSFORMATION TO CONVEX OPTIMIZATION

The temperature of the working fluid, the density of the working fluid, the entropy production, etc. are connected with each other through the equations of the detailed model [1]. Here we call these dependences the inner constraints. In order to solve Problem 1′ we break some of the inner constraints, and consider the temperature $T(t,\xi) > 0$ and the global entropy production $\sigma(t) \geq 0$ as additional control variables. If we maximize efficiency with respect to these additional control variables, taking into account the periodicity of energy and entropy, the result will give us the desired upper bound. We thus obtain the averaged optimal control Problem 1″: to maximize $-Q_H$ with constraints $\tau\mathcal{P} = \tau\mathcal{P}^0$, Eq. (6), using the additional control $T > 0$, $\sigma \geq 0$. In order to solve this problem we multiply the two integral constraints by two scalar Lagrange multipliers, l and λ, and add them to $-Q_H$. After this we consider an unconstrained, averaged optimization problem: to maximize

$$-Q_H+l\tau(\mathcal{P}-\mathcal{P}^0)+\lambda\int_0^\tau\left[v_H\int_{A_H}\frac{q_H}{T}da+v_L\int_{A_L}\frac{q_L}{T}da+\sigma(t)\right]dt$$

with respect to the additional control $T>0$, $\sigma\geq 0$.

Let

$$\phi_1(l,\lambda)=\max_{T>0}\int_0^\tau\left[v_H\int_{A_H}q_H(l-1+\lambda/T)da+v_L\int_{A_L}q_L(l+\lambda/T)da\right]dt\ . \quad (7)$$

From the definition (7) the inequality for heat follows:

$$-Q_H\leq\phi_1(l,\lambda)-l\tau\mathcal{P}^0-\lambda\int_0^\tau\sigma(t)dt\ . \quad (8)$$

Inequality (8) is valid for all real l, λ. However, the right-hand side of Eq. (7) is bounded for Newtonian heat-exchange laws only for $\lambda<0$, $l>1$. Consequently, we choose these parameters to satisfy $\lambda<0$, $l>1$. For these Lagrange multipliers $\lambda\int_0^\tau\sigma(t)dt\leq 0$ and we may omit this term from the inequality (8), in using it to get the upper bound. We now get a series of upper bounds depending upon $\lambda<0$, $l>1$ and our next step is to find the best bound among them.

The final optimizational problem 1‴ is to minimize $\phi_1(l,\lambda)-l\tau\mathcal{P}^0$ with constraints $l>1$, $\lambda<0$. Let $\hat{l}$, $\hat{\lambda}$ be the solution of this problem. Then for a given admissible external control, $Q_H\geq\hat{l}\tau\mathcal{P}^0-\phi_1(\hat{l},\hat{\lambda})$. If the right-hand part of this inequality is positive, we get the estimate for efficiency in Problem 1:

$$\eta_1\leq\frac{\tau\mathcal{P}^0}{\hat{l}\tau\mathcal{P}^0-\phi_1(\hat{l},\hat{\lambda})}\ . \quad (9)$$

Remark 2. Estimate (9) is true also for any l, λ: $l\tau\mathcal{P}^0-\phi_1(l,\lambda)>0$. It is easy to see that the function $\phi_1(l,\lambda)$ is convex.

For $l>1$, $\lambda<0$ and Newtonian heat-exchange laws we can carry out maximization in Eq. (7) analytically. The result is $\hat{T}=[-\lambda T_H/(l-1)]^{1/2}$ for ξ varying on the surface A_H, $\hat{T}=(-\lambda T_L/l)^{1/2}$ for ξ varying on the surface A_L. Substituting these functions into Eq. (7), we get

$$\phi_1(l,\lambda)=\int_0^\tau\left[\int_{A_H}v_H\alpha_H\{[T_H(l-1)]^{1/2}-\sqrt{-\lambda}\}^2da+\int_{A_L}v_L\alpha_L[(T_Ll)^{1/2}-\sqrt{-\lambda})]^2da\right]dt\ . \quad (10)$$

Problem 2 is transformed into the final convex optimization Problem 2‴: to minimize $\phi_2(l,\lambda)-lQ_H^0$ with constraints $l>-1$, $\lambda<0$, where

$$\phi_2(l,\lambda)=\max_{T>0}\int_0^\tau\left[v_H\int_{A_H}q_H(l+1+\lambda/T)da+v_L\int_{A_L}q_L(1+\lambda/T)da\right]dt\ . \quad (11)$$

For Newtonian heat-exchange laws

$$\phi_2(l,\lambda)=\int_0^\tau\left[\int_{A_H}v_H\alpha_H\{[T_H(l+1)]^{1/2}-\sqrt{-\lambda}\}^2da+\int_{A_L}v_L\alpha_L[(T_L)^{1/2}-\sqrt{-\lambda}]^2da\right]dt\ . \quad (12)$$

All intermediate steps are the same as for Problem 1.

Let $\hat{l}$, $\hat{\lambda}$ be the solution of the Problem 2‴. Then for a given admissible control $\tau\mathcal{P}\leq\phi_2(\hat{l},\hat{\lambda})-\hat{l}Q_H^0$. From this inequality the estimate of efficiency η_1 follows:

$$\eta_1\leq\frac{\phi_2(\hat{l},\hat{\lambda})-\hat{l}Q_H^0}{Q_H^0}\ . \quad (13)$$

Problem 3 is transformed to the final convex optimization Problem 3‴: to minimize $\phi_3(l,\lambda)-l\tau\mathcal{P}^0$ with constraints $l>1$, $\lambda<0$, where

$$\phi_3(l,\lambda)=\max_{T>0}\int_0^\tau\left[v_H\int_{A_H}q_H[l-\mathrm{sg}(q_H)+\lambda/T]da+v_L\int_{A_L}q_L(l+\lambda/T)da\right]dt\ . \quad (14)$$

Here $\mathrm{sg}(x)=1$ when $x>0$, $\mathrm{sg}(x)=0$ when $x\leq 0$. For Newtonian heat-exchange laws $\phi_3(l,\lambda)=\psi_H(l,\lambda)+\psi_L(l,\lambda)$, where

$$\psi_H=\int_0^\tau\int_{A_H}v_H\alpha_H(\{T_H[l-\mathrm{sg}(T_H-T^0)]\}^{1/2}-\sqrt{-\lambda})^2\times[\mathrm{sg}(l-1+\lambda/T_H)+\mathrm{sg}(T^0-T_H)]da\,dt\ ,$$
$$\psi_L=\int_0^\tau\int_{A_L}v_L\alpha_L[(T_Ll)^{1/2}-\sqrt{-\lambda}]^2da\,dt\ . \quad (15)$$

Here $T^0=-\lambda/l$.

Let $\hat{l}$, $\hat{\lambda}$ be the solution of this problem. Then for a given admissible control $Q_H^+\geq\hat{l}\tau\mathcal{P}^0-\phi_3(\hat{l},\hat{\lambda})$. If $\hat{l}\tau\mathcal{P}^0-\phi_3(\hat{l},\hat{\lambda})>0$, we get the estimate of efficiency in Problem 3:

$$\eta_2\leq\frac{\tau\mathcal{P}^0}{\hat{l}\tau\mathcal{P}^0-\phi_3(\hat{l},\hat{\lambda})}\ . \quad (16)$$

Remark 3. Estimate (16) is true also for any l, λ: $l\tau\mathcal{P}^0-\phi_3(l,\lambda)>0$.

The problem of finding an upper bound $\tilde{\mathcal{P}}_{\max}$ for the average power output, analyzed in Ref. [1] for the Newtonian heat-exchange laws, is reduced in the case of general heat-exchange laws to the one-dimensional convex optimization problem

$$\tau\tilde{\mathcal{P}}_{\max}=\min_{\lambda<0}\phi_2(0,\lambda)\ . \quad (17)$$

IV. ANALYTICAL EXPRESSIONS FOR MAXIMUM EFFICIENCY

Let us first recall the analytical expression for upper bound of the power output from Ref. [1]. With notations

$$\overline{f_H}=\frac{1}{\tau}\int_0^\tau\left[\frac{1}{a_H}\int_{A_H(t)}f_H(t,\xi,\lambda)da\right]dt,$$

$$\overline{f_L}=\frac{1}{\tau}\int_0^\tau\left[\frac{1}{a_L}\int_{A_L(t)}f_L(t,\xi,\lambda)da\right]dt\ ,$$

this upper bound for Newtonian heat-exchange laws has the form

$$\tilde{\mathcal{P}}_{\max}=\overline{v_H a_H \alpha_H[(T_H)^{1/2}-(-\hat{\lambda})^{1/2}]^2}+\overline{v_L a_L \alpha_L[(T_L)^{1/2}-(-\hat{\lambda})^{1/2}]^2}\ , \tag{18}$$

with

$$(-\hat{\lambda})^{1/2}=[\overline{v_H a_H \alpha_H (T_H)^{1/2}}+\overline{v_L a_L \alpha_L (T_L)^{1/2}}]/(\gamma_H+\gamma_L)\ ,$$

$$\gamma_H=\overline{v_H a_H \alpha_H},\quad \gamma_L=\overline{v_L a_L \alpha_L}\ .$$

For reservoirs at constant temperatures, expression (18) has the form [1]

$$\tilde{\mathcal{P}}_{\max}=\frac{\gamma_H\gamma_L}{\gamma_H+\gamma_L}[(T_H)^{1/2}-(T_L)^{1/2}]^2\ . \tag{19}$$

The upper bound of the power output (18) is a functional (a function whose arguments are functions) of the reservoir temperatures $T_H(t,\xi)$ and $T_L(t,\xi)$, so we could write it using T_H and T_L as arguments: $\tilde{\mathcal{P}}_{\max}=\mathcal{P}_{\max}(T_H,T_L)$. Using Eq. (18), we could reduce Problems 1‴ and 2‴ (for linear heat-exchange laws) to one-dimensional optimizations, so the corresponding Problem for 1‴ is to minimize $\psi_1(l)=\tilde{\mathcal{P}}_{\max}((l-1)T_H, lT_L)-l\mathcal{P}^0$ with the constraint $l>1$. The corresponding problem for 2‴ is to minimize $\psi_2(l)=\tau\tilde{\mathcal{P}}_{\max}((l+1)T_H,T_L)-lQ_H^0$ with the constraint $l>-1$.

For constant-temperature reservoirs, these problems can be solved analytically. Solution $\hat{l}_1$ of the first problem has the form

$$\hat{l}_1=\{1+[1+1/(z^2-1)]^{1/2}\}/2\ ,$$

where

$$z=(T_H+T_L-\mathcal{P}^0/c)/[2(T_H T_L)^{1/2}]$$

$$c=\gamma_H\gamma_L/(\gamma_H+\gamma_L)\ .$$

Corresponding estimates are

$$\begin{aligned}Q_H&\ge-\tau\psi_1(\hat{l})\\&=\tau c(T_H T_L)^{1/2}[(T_H/T_L)^{1/2}-z-(z^2-1)^{1/2}]\ ,\\ \eta_1&\le\tilde{\eta}_{\max}=\frac{\mathcal{P}^0}{-\psi_1(\hat{l})}=1-\left[\frac{T_L}{T_H}\right]^{1/2}[z-(z^2-1)^{1/2}]\ .\end{aligned} \tag{20}$$

The inequality $z\ge1$ is equivalent to the inequality $\mathcal{P}^0\le\tilde{\mathcal{P}}_{\max}$. It is easy to see that $\tilde{\eta}_{\max}\to1-(T_L/T_H)^{1/2}$ when $\mathcal{P}^0\to\tilde{\mathcal{P}}_{\max}$; $\tilde{\eta}_{\max}\to1-T_L/T_H$ when $\mathcal{P}^0\to0$. The estimate of efficiency in Eq. (20) is valid for both the distributed-parameter model and the lumped-parameter model. In the last case instead of integrating over contact surfaces A_H, A_L in Eq. (18) to calculate γ_H, γ_L we simply multiply by areas a_H, a_L correspondingly. We thus obtained the famous Curzon-Ahlborn result [3] for efficiency at maximum power point: $\eta_1\le1-(T_L/T_H)^{1/2}$. Here we have proved the applicability of this upper bound for a heat engine whose working fluid is viscous and not necessarily uniform in temperature. For lumped-parameter models with constant a_H, a_L, α_H, α_L, such that $a_H\alpha_H=a_L\alpha_L$, the expression for efficiency, which is equivalent to Eq. (20), was published in Ref.[4].

Solution of the second one-dimensional problem has the form

$$\hat{l}_2=T_H T_L/[T_H-Q_H^0/(\tau c)]^2-1\ .$$

Corresponding estimates are

$$\tau\mathcal{P}\le\psi_2(\hat{l})=Q_H^0-\frac{T_L}{T_H}\frac{Q_H^0}{1-Q_H^0/(\tau c T_H)}$$

and

$$\eta_2\le\frac{\psi_2(\hat{l})}{Q_H^0}=1-\frac{T_L}{T_H}\frac{1}{1-Q_H^0/(\tau c T_H)}\ , \tag{21}$$

where $c=\gamma_H\gamma_L/(\gamma_H+\gamma_L)$; γ_H, γ_L are defined in Eq. (18). Inequality (21) is valid for both distributed- and lumped-parameter models. For a lumped-parameter model with constant a_H, a_L, α_H, α_L such that $a_H\alpha_H=a_L\alpha_L$ this estimate was published in Ref. [2].

V. NUMERICAL EXPERIMENTS AND DISCUSSION

For the lumped-parameter model with Newtonian heat exchange, inequality (9) could not be improved for $v_H(t)=0$ when $t>t_H$ and $v_L(t)=0$ when $t\le t_H$, $0<t_H<\tau$, because, in this case

$$\hat{T}(t)=[-\hat{\lambda}T_H(t)/(\hat{l}-1)]^{1/2}$$

when $t\le t_H$, and $\hat{T}(t)=[-\hat{\lambda}T_L(t)/\hat{l}]^{1/2}$ when $t>t_H$, give the optimal process for Problem 1. Indeed, for such v_H, v_L the estimate (9) is uniform (it is independent of any admissible control) so long as $v_H(t)=1$ when $0\le t\le t_H$, $v_H(t)=0$ when $t_H\le t\le\tau$, $v_L(t)=0$ when $0\le t\le t_H$, and $v_L(t)=1$ when $t_H<t\le\tau$. The efficiency of the process described above is equal to the value of this estimate. The same is true for inequality (13). The corresponding optimal process for Eq. (9) is

$$\hat{T}(t)=[-\hat{\lambda}T_H(t)/(\hat{l}+1)]^{1/2}$$

TABLE I. The ratio of the average power output for the different values of the temperature variation ΔT to the average power output for zero-temperature variation (constant-temperature reservoirs).

$\Delta T/T_H^0$	$\tilde{\mathcal{P}}_{\max}(\Delta T)/\tilde{\mathcal{P}}_{\max}(0)$
0.000	1.000
0.200	1.029
0.400	1.121
0.600	1.287

TABLE II. The ratio of the efficiency η_1 for the different values of the temperature variation ΔT to the efficiency η_1 for zero-temperature variation.

$\Delta T/T_H^0$	$\eta_1(\Delta T)$	$\eta_1(\Delta T)/\eta_1(0)$	η_c
0.0000	0.6412	1.0000	0.7557
0.2000	0.6599	1.0292	0.7964
0.4000	0.7157	1.1161	0.8255
0.6000	0.8127	1.2674	0.8473

TABLE III. The ratio of the efficiency η_2 for the different values of the temperature variation ΔT to the efficiency η_2 for zero-temperature variation.

$\Delta T/T_H^0$	$\eta_2(\Delta T)$	$\eta_2(\Delta T)/\eta_2(0)$
0.0000	0.6412	1.0000
0.2000	0.6599	1.0292
0.4000	0.7099	1.1071
0.6000	0.7568	1.1800

when $t \le t_H$, $\hat{T}(t)=[-\hat{\lambda}T_L(t)]^{1/2}$ when $t>t_H$, and for Eq. (13), is $\hat{T}(t)=[-\hat{\lambda}T_H(t)]^{1/2}$ when $t\le t_H$, $\hat{T}(t)=[-\hat{\lambda}T_L(t)]^{1/2}$ when $t>t_H$.

We computed the maximum power output and the maximum efficiency for a given power output for the lumped-parameter model with an oscillating high-reservoir temperature. In this model $T_H(t)=T_H^0+\Delta T\sin(4\pi t)$, $T_L(t)=T_L^0$, $\tau=1$ sec, $t_H=0.5\tau$, $v_H(t)=1$ when $0\le t\le t_H$, $v_H(t)=0$ when $t_H<t\le\tau$, $v_L(t)=0$ when $0\le t\le t_H$, $v_L(t)=1$ when $t_H<t\le\tau$, $\alpha_H a_H=100$ W/K, $\alpha_L a_L=100$ W/K, $T_H^0=1200$ K, $T_L^0=293.15$ K.

Table I shows the values of power output $\mathcal{P}_{\max}$ for different values of ΔT, $\tilde{\mathcal{P}}_{\max}(0)$ is the value of $\tilde{\mathcal{P}}_{\max}$ evaluated at $\Delta T=0$.

Computations were carried out using expressions (18) and (19). Integrals in Eq. (18) were computed using the Simpson quadrature formula. Results presented in Table I show the growth of the power output with the growth of the amplitude of temperature variation of the high-temperature reservoir.

Table II shows the growth with increasing ΔT of maximum efficiency η_1 for given $\mathcal{P}^0=0.8\tilde{\mathcal{P}}_{\max}(0)$; $\eta_1(0)$ is the value of η_1 evaluated at $\Delta T=0$. For comparison we also give values of the Carnot efficiency $\eta_c=1-T_L^0/(T_H^0+\Delta T)$.

Computations were carried out via the method of golden section [5] applied for the one-dimensional final problem. For $\Delta T=0$ we used expressions (20) to check the algorithm. To double check, we also computed results for the two-dimensional Problem 1‴ via a convex minimization procedure [5]. The results are, of course, the same.

Table III shows a growth of maximum efficiency η_2 for given $\mathcal{P}^0=0.8\tilde{\mathcal{P}}_{\max}(0)$ with the growth of ΔT.

Results of Table III were computed via Powell's convex minimization procedure [5] applied to Problem 3‴.

Similar efficiency growth was reported in Ref. [6] for the harmonically driven volume of the heat engine.

Comparison of Tables II and III shows that for time- and space-varying reservoir temperatures, definitions (3) and (5) for the efficiency could lead to different estimated values of the efficiency. This conclusion is the reason for presenting the results of computations for the Newtonian heat-exchange law and the range of temperature variation up to $\Delta T/T_H=0.6$. In reality, for such large $\Delta T/T_H$ it would be appropriate to take into account radiative heat transfer. One could do it solving convex optimization problems of the kind formulated in Sec. III for general heat-exchange laws.

With the help of the numerical solution of the Problem 3‴ [that is, using the efficiency definition (5)] one could compute the upper bounds for the power and efficiency of real Stirling engines. For many of them it is reasonable to assume that $v_H(t)=1$, $v_L(t)=1$ for $0\le t\le\tau$. Information which is needed for such computations is the shape of contact surfaces A_H and A_L, heat-exchange laws $q_H(\xi,T,T_H)$ and $q_L(\xi,T,T_L)$, and temperatures of reservoirs $T_H(t,\xi)$ and $T_L(t,\xi)$. From these data, the upper bound for the power should be calculated via Eq. (17), and the upper bound for the efficiency with fixed average power output should be calculated via Eq. (16). Of course, the actual power and efficiency of a real Stirling engine could be much less than the upper bounds presented here. One of the likely reasons for such differences would be losses in the thermal sponge. Nevertheless, these bounds could be used as an universal yardstick when we need to compare different types of engines and also as a basis for further improved estimates which would take into account every term in the expression for the entropy production.

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