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Power output from an irreversible heat engine with a nonuniform working fluid

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The upper bound is calculated for the average power output of an irreversible heat engine whose working fluid is viscous and not necessarily uniform in temperature. The calculation is done via the generalized formalism of finite-time thermodynamics. The working fluid of the heat engine is described by partial-differential equations containing parameters in the boundary conditions. With some restrictions on parameters, the estimated power coincides with the optimal power output for the classical lumped-parameter (uniform-T) irreversible Carnot engine.

I. INTRODUCTION

In this paper an upper bound on average power output of an irreversible heat engine is obtained. The engine is "Carnot-like": it consists of the working fluid, the machine, and two heat reservoirs. The working fluid of the engine is described by viscous, nonisothermal gas dynamics equations. Boundary conditions for these equations contain switching functions $v_H(t)$ and $v_L(t)$. These functions of time t regulate the finite-rate heat transfer between the working fluid and two heat reservoirs. Physically, the functions v_H and v_L imply the existence of two switching devices between reservoirs and the working fluid. When $0 < v_H(t) \leq 1$ the high-temperature reservoir and the working fluid are in contact and exchange energy; when $v_H(t)=0$ there is no exchange between them. When $0 < v_L(t) \leq 1$ the low-temperature reservoir and the working fluid are in contact and exchange energy; when $v_L(t)=0$ there is no exchange between them. The shape of the working-fluid boundary may vary in time. The initial and final states of the working fluid in the distributed parameter model need not be equilibrium states so a problem arises of the definition of an analogue of a periodic process. Indeed, even if we start at $t=0$ from the initial equilibrium state with zero velocity, constant temperature, and constant pressure, and smoothly change the shape of the working fluid, we shall find that for $t>0$ the velocity is nonzero, and pressure and temperature are not uniform in space. These imply that for any finite time $t>0$ the working fluid must be in a nonequilibrium state. We define a weakly periodic process as a process for which $E(0)=E(\tau)$ and $S(0)=S(\tau)$, where $E(t)$ and $S(t)$ are the total energy and the total entropy of the working fluid; $\tau>0$ is the period of the process. For weakly periodic solutions of the equations describing the working fluid, we show that the time-averaged power output is lower than or equal to

$$\hat{\mathcal{P}}_{\max}=\frac{a_H\alpha_H a_L\alpha_L}{(\sqrt{a_H\alpha_H}+\sqrt{a_L\alpha_L})^2}(\sqrt{T_H}-\sqrt{T_L})^2 ,$$

where $T_H > T_L$ are the constant temperatures of the system's two reservoirs, constants a_H and a_L are areas of contact with hot and cold reservoirs, respectively, and $\alpha_H>0$ and $\alpha_L>0$ are the constant heat-transfer coefficients. This result is obtained as a corollary of a more general estimation of power output in the problem with time- and space-dependent reservoir temperatures $T_H(t,\xi)$ and $T_L(t,\xi)$, space-dependent heat-transfer coefficients $\alpha_H(\xi)$ and $\alpha_L(\xi)$, and time-dependent areas $a_H(t)$ and $a_L(t)$ of the contact surfaces $A_H(t)$ and $A_L(t)$. The value of $\hat{\mathcal{P}}_{\max}$ is the same as the value of maximum average power of the lumped-parameter irreversible Carnot heat engine, first obtained by Curzon and Ahlborn.[1]

Successful applications of finite-time thermodynamics now include the analysis of Carnot-based heat engines,[1–12] general endoreversible heat engines,[13–16] internal-combustion engines,[17] chemical-reaction systems,[18] separation processes,[19,20] light-driven engines,[21] electrochemical systems,[22] fuel synthesis,[23] and solar-driven heat engines.[24] In these publications, simplified mathematical descriptions of the processes by ordinary differential equations (lumped-parameter models) were used, together with the help of mass, energy, and entropy balances. In this work we describe the simplest heat engine using the more detailed apparatus of partial differential equations; thus, a distributed parameter model is used. We obtain estimates of power output using entropy balance, breaking of some constraints by introducing the set of variables treated as controls, and Lagrange multipliers. This method is applicable both to lumped-parameter models and to distributed-parameter models. For a better understanding, the estimate is obtained first for a lumped-parameter model with time-varying temperatures T_H and T_L and then for a distributed-parameter model. The structure of Secs. II–IV is the following: the beginning of each section is devoted to the lumped-parameter model, the middle part of each section is devoted to the distributed-parameter model, and the final part is a short discussion concerning the physical meaning of the previous two parts.

II. DESCRIPTION OF THE MODELS

The lumped-parameter irreversible Carnot heat engine is described by the ordinary differential equation

$$\frac{dE}{dt}=v_H a_H\alpha_H(T_H-T)+v_L a_L\alpha_L(T_L-T)-p\frac{dV}{dt} , \quad (1)$$

where $T=T(E,V)$ and $p=p(E,V)$ are the temperature and the pressure of the working fluid, respectively (these equations of state are assumed given); E and V are, respectively, the energy and the volume of the working fluid; functions $V(t)$, $v_H(t)$, and $v_L(t)$ are given. These functions are called admissible parameters, if $V(0)=V(\tau)$, $0\le v_H(t)\le 1$, and $0\le v_L(t)\le 1$, and there exists a solution of (1) satisfying the condition $E(0)=E(\tau)$.

The average power output of this engine has the form $\mathcal{P}=(1/\tau)\int_0^\tau p(dV/dt)dt$. Here $\tau>0$ is the period. Using $E(0)=E(\tau)$ and (1) we get

$$\mathcal{P}=\frac{1}{\tau}\int_0^\tau [v_H a_H \alpha_H(T_H-T)+v_L a_L \alpha_L(T_L-T)]dt \ . \tag{2}$$

Problem 1: to find an upper bound for the average power (2) based on the solution of (1) with given admissible parameters.

The distributed-parameter irreversible heat engine under consideration is described by the conservation equations for energy, mass, and momentum densities:[25]

$$\frac{\partial \varepsilon}{\partial t}+\frac{\partial}{\partial \xi_j}[(\varepsilon+p)-U_{ij}u_i+q_j]=0 \ ,$$
$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial \xi_j}(\rho u_j)=0 \ , \tag{3}$$
$$\frac{\partial \pi_i}{\partial t}+\frac{\partial}{\partial \xi_j}(\pi_i u_j+p\delta_{ij}-U_{ij})=0 \ ,$$
$$i,j=1,2,3, \quad \xi\in\Omega(t) \ ,$$

and the boundary conditions

$$[(\varepsilon+p)(u_j-D_j)-U_{ij}u_i+q_j]\nu_j + v(t,\xi)\alpha(\xi)[T_R(t,\xi)-T]=0 \ ,$$
$$\rho(u_j-D_j)\nu_j=0 \ , \tag{4}$$
$$[\pi_i(u_j-D_j)-U_{ij}]\nu_j=0, \quad i,j=1,2,3, \quad \xi\in\partial\Omega(t) \ ,$$

where

$$U_{ij}=\lambda^0\frac{\partial u_l}{\partial \xi_l}\delta_{ij}+\mu^0\left[\frac{\partial u_i}{\partial \xi_j}+\frac{\partial u_j}{\partial \xi_i}\right]$$

is the viscous stress tensor, and

$$q_j=-\kappa^0\frac{\partial T}{\partial \xi_j}$$

is Fourier's heat-transfer law. Here and further, summation is understood over the repeated indices. In (3), $\varepsilon(t,\xi)$, $\rho(t,\xi)$, and $\pi_i(t,\xi)$ are the total-energy density, the mass density, and the momentum density, respectively. Functions $u_i(t,\xi)=\pi_i/\rho$, $i=1,2,3$, are velocity components; T and p are the temperature and the pressure; λ^0 and μ^0 are given viscosity coefficients such that $3\lambda^0+2\mu^0>0$; $\kappa^0>0$ is the thermal conductivity; $\delta_{ij}=1$ when $i=j$ and $\delta_{ij}=0$ when $i\neq j$, $i,j=1,2,3$. The equation system (3) is closed with the help of equations of state $T=T(e,\rho)$, $p=p(e,\rho)$, and $e=\varepsilon-\pi_i\pi_i/(2\rho)$, where e is the internal energy density. In (4), $D_j(t,\xi)$ are the given boundary velocity components, $\partial\Omega(t)$ is the boundary of the working fluid, $\nu(t,\xi)$ is the outer normal vector at the point $\xi\in\partial\Omega(t)$, and $v(t,\xi)$ is a switching function regulating heat transfer between the working fluid and two heat reservoirs. This function is assumed to consist of only the time-dependent switching function $v_H(t)$ at the surface of contact with the high-temperature reservoir, only the time-dependent switching function $v_L(t)$ at the surface of contact with the low-temperature reservoir, and zero at the other points of the working-fluid boundary. In more precise language $v(t,\xi)=v_H(t)$ when ξ is on $A_H(t)$, which is the high-temperature part of the boundary $\partial\Omega(t)$, $v(t,\xi)=v_L(t)$ when ξ is on $A_L(t)$, which is the low-temperature part of the boundary, and $v(t,\xi)=0$ when ξ is on neither of the boundaries corresponding to contact with the reservoirs. Functions $T_R(t,\xi)$ and $\alpha(\xi)$ have the form $T_R(t,\xi)=T_H(t,\xi)$ when ξ is on $A_H(t)$, $T_R(t,\xi)=T_L(t,\xi)$ when ξ is on $A_L(t)$, $\alpha(\xi)=\alpha_H(\xi)$ when ξ is on $A_H(t)$, $\alpha(\xi)=\alpha_L(\xi)$ when ξ is on $A_L(t)$, where $T_H(t,\xi)$ and $T_L(t,\xi)$ are given temperatures of the reservoirs, and $\alpha_H(\xi)>0$ and $\alpha_L(\xi)>0$ are heat-transfer coefficients.

Functions $D_j(t,\xi)$ $(j=1,2,3)$, $v_H(t)$, and $v_L(t)$ are called admissible parameters if $0\le v_H(t)\le 1$, $0\le v_L(t)\le 1$, and a solution exists for (3) and (4), satisfying the conditions $E(0)=E(\tau)$, $S(0)=S(\tau)$, and $\rho(t,\xi)>0$. Here $E(t)=\int_{\Omega(t)}\varepsilon(t,\xi)d\xi$ is the total energy, $S(t)$ is the total entropy of the working fluid, and the integral is taken over the volume of the working fluid $\Omega(t)$.

The average power outputs has the form

$$\mathcal{P}=\frac{1}{\tau}\int_0^\tau\left[\int_{\partial\Omega(t)} pD_j\nu_j da\right]dt \ .$$

Here $\tau>0$ is the weak period. Using the condition $E(0)=E(\tau)$ and (3) and (4), we get

$$\mathcal{P}=\frac{1}{\tau}\int_0^\tau\left[\int_{A_H} v_H\alpha_H(T_H-T)da + \int_{A_L} v_L\alpha_L(T_L-T)da\right]dt \ . \tag{5}$$

Problem 2: to find an upper bound for the average power (5) based on the solution of (3) and (4), with given admissible parameters.

Consider the illustrative example of a working fluid in a cylinder with the axis oriented along coordinate ξ_1; the motion of a piston is $x(t)$. We assume that the bottom of the cylinder is in contact with the high-temperature reservoir; all other points of the cylinder's boundary contact the low-temperature reservoir (the environment). The heated surface $A_H(t)$ has constant area $a_H=\pi r^2$, where r is the radius of the cylinder. The rest of the system's surface $A_L(t)$ varies in time. Its area is $a_L(t)=\pi r^2+2\pi r x(t)$. The working fluid occupies the volume $\xi=(\xi_1,\xi_2,\xi_3)$: $0\le\xi_1\le x(t)$, $\xi_2^2+\xi_3^2\le r^2$. The velocity component at the moving boundary of the piston is $D_1(t,\xi)=dx/dt$ for $\xi_1=x(t)$, $\xi_2^2+\xi_3^2\le r^2$. This function is zero at other points of the working-fluid boundary $\partial\Omega(t)$. The boundary velocity components D_2 and D_3 are zero. If the thickness of the walls of our engine were to vary in space, the heat-transfer coefficients $\alpha_H(\xi)$ and $\alpha_L(\xi)$ would vary accordingly.

Lumped-parameter or system-averaged models are applicable to real engines when the speeds of their pistons are low compared with the speed of sound in the working fluid. The distributed-parameter model is required for real engines when the speed of the piston is comparable to the speed of sound. The most important type of real engine for which this formulation may be useful is, of course, the Stirling cycle engine.

We shall not solve these first two problems directly, but transform them to a more convenient form with the help of entropy balance.

III. ENTROPY BALANCE AND CONSTRAINT BREAKING

The entropy of the working fluid in the lumped-parameter model is a given function of energy, volume, and the amount of material: $S=S(E,V,N)$.[26] Differentiating this function in the solution of (1) and using the equations of state $\partial S/\partial E=1/T$ and $\partial S/\partial V=p/T$ and the facts that $N(t)=\text{const}$ and $T>0$, we get the entropy balance

$$\frac{dS}{dt}=\frac{v_H a_H \alpha_H(T_H-T)+v_L a_L \alpha_L(T_L-T)}{T} \ . \tag{6}$$

We express T in (6) as a function of S, V, and N, as is usual in energy representation.[26] With this, Eq. (6) becomes closed; i.e., we have one equation for one unknown function $S(t)$. From $E(0)=E(\tau)$ and $V(0)=V(\tau)$, it follows that $S(0)=S(\tau)$. Instead of solving Problem 1 directly we shall solve the heirarchy of problems:

Problem 1′: find an upper bound of (2) based on the solution of (6) with given admissible parameters.

Problem 1″: maximize (2) subject to the constraint that the integral of (6) over a cycle be zero, provided the temperature T is always positive.

In order to get the solution of Problem 1′, which is equivalent to Problem 1, we break the constraint $T=T(S,V,N)$ in (6) and consider temperature $T(t)>0$ as a control. The right sides of (6) as well as of (2) are now independent of the state variable S and we obtain the so-called averaged optimal control Problem 1″.[5,7] Explicitly the constraint is

$$\frac{1}{\tau}\int_0^\tau \frac{v_H a_H \alpha_H(T_H-T)+v_L a_L \alpha_L(T_L-T)}{T}dt=0 \ . \tag{7}$$

The solution of the optimization Problem 1″ or its upper bound solves Problems 1 and 1′, because the real solution $T(t)$ of (6) for given admissible parameters satisfies (7) and is positive, so it belongs to the set of admissible controls.

The entropy of the working fluid in the distributed-parameter model is determined by a function $s_0=s_0(e,n)=S(E,V,N)/V$, where e is the internal energy density E/V, and n is the molecular number density N/V. In the solution $\varepsilon(t,\xi)$, $\rho(t,\xi)$, $\pi(t,\xi)$ of (3) and (4), the entropy of the working fluid has the form

$$S(t)=\int_{\Omega(t)} s_0\left[\varepsilon-\frac{\pi_i\pi_i}{2\rho},\frac{\rho}{m^0}\right]d\xi \ ,$$

where m^0 is the mass of one molecule of working fluid. Differentiating the total entropy in the solution of (3) and (4) and using the equations of state $1/T=\partial s_0/\partial e$, $-\mu/T=\partial s_0/\partial n$, and $p/T=s_0-e/T+(\mu/T)n$ and the fact that $T>0$, we get the entropy balance

$$\frac{dS}{dt}=\int_{A_H}\frac{v_H\alpha_H(T_H-T)}{T}da+\int_{A_L}\frac{v_L\alpha_L(T_L-T)}{T}da+\sigma(t) \ , \tag{8}$$

where

$$\sigma=\int_{\Omega(t)}\left\{\sum_{i=1}^{3}\left[\frac{\kappa^0}{T^2}\left[\frac{\partial T}{\partial \xi_i}\right]^2\right]+\frac{\lambda^0}{T}\left[\frac{\partial u_i}{\partial \xi_i}\right]^2+\frac{\mu^0}{2T}\sum_{i,j=1}^{3}\left[\frac{\partial u_i}{\partial \xi_j}+\frac{\partial u_j}{\partial \xi_i}\right]^2\right\}d\xi \ .$$

The function $\sigma(t)\geq 0$ is the total entropy production in the working fluid. Instead of solving Problem 2 directly we shall solve a second little heirarchy.

First we address Problem 2′: find an upper bound of (5) based on the solutions of (8) with given admissible parameters. Equation (8) is not closed: we have one integral equation for many unknown functions, so it may have many solutions for given admissible parameters. The upper limit of Problem 2′ solves Problem 2, because the real solution of (3) and (4) with admissible parameters satisfies (8). Note that Problems 2 and 2′ are not equivalent.

In order to get the solution of the Problem 2′ we break the constraint $T=T(\varepsilon,\rho,\pi)$ and consider temperature $T=T(t,\xi)>0$ as the control and $\sigma(t)\geq 0$ as a given function. We shall get an estimation of (5) by solving the averaged optimal control Problem 2″: we maximize (5) with constraint

$$\frac{1}{\tau}\int_0^\tau\left[\int_{A_H}\frac{v_H a_H \alpha_H(T_H-T)}{T}da+\int_{A_L}\frac{v_L a_L \alpha_L(T_L-T)}{T}da\right]dt=\delta \tag{9}$$

over the set of admissible control $T>0$. Here

$$\delta=-\frac{1}{\tau}\int_0^\tau \sigma(t)dt\leq 0 \ .$$

The constraint (9) depends on the nonpositive parameter δ. The upper bound in Problem 2″, which is valid for every $\delta\leq 0$, solves Problems 2 and 2′.

In order to get estimates of power output we should now solve averaged optimal control problems. These problems are much easier than general optimal control problems. With some additional constraints on admissible parameters these problems could be solved analytically. But here we again do not solve them directly. Instead, we follow the still easier course and get upper limits for these problems with the help of Lagrange multipliers. The surprising fact is that final estimations are not rough.

IV. AVERAGED PROBLEMS AND LAGRANGE MULTIPLIERS

Let $\mathcal{P}_{\max}=\sup\mathcal{P}$ over the set of admissible controls of the corresponding problems 1″ and 2″.

For Problem 1″, the lumped-parameter model,

$$\mathcal{P}_{\max}\le\max_{T>0}\left\{\frac{1}{\tau}\int_0^\tau\left[v_H(t)a_H\alpha_H(T_H-T)\left[1+\frac{\lambda}{T}\right]+v_L(t)a_L\alpha_L(T_L-T)\left[1+\frac{\lambda}{T}\right]\right]dt\right\}. \tag{10}$$

In (10), $\lambda<0$ is a given Lagrange multiplier, and $T(t)>0$ is the control. Here we again break the constraint (7) in Problem 1″ and get a range of estimates (10), depending on λ. Maximizing the first term of the integrand with T we get the optimal argument

$$\hat{T}_1=\sqrt{-\lambda T_H(t)}$$

and the optimal value of this term

$$f_H(t,\lambda)=v_H(t)a_H(t)\alpha_H[\sqrt{T_H(t)}-\sqrt{-\lambda}]^2\ .$$

From the same maximization of the second term in (10) we get

$$\hat{T}_2=\sqrt{-\lambda T_L(t)}$$

and the optimal value of the term

$$f_L(t,\lambda)=v_L(t)a_L(t)\alpha_L[\sqrt{T_L(t)}-\sqrt{-\lambda}]^2\ .$$

It is evident that

$$\mathcal{P}_{\max}\le\frac{1}{\tau}\int_0^\tau[f_H(t,\lambda)+f_L(t,\lambda)]dt\ .$$

Minimizing the right part of this inequality with the variable $\lambda<0$ we get the condition for the optimum $\hat{\lambda}$ and the limiting value of the power $\tilde{\mathcal{P}}_{\max}$. The solution has the form

$$\begin{aligned}\tilde{\mathcal{P}}_{\max}&=\overline{v_Ha_H\alpha_H[\sqrt{T_H}-(-\hat{\lambda})^{1/2}]^2}+\overline{v_La_L\alpha_L[\sqrt{T_L}-(-\hat{\lambda})^{1/2}]^2}\ ,\\ \sqrt{-\hat{\lambda}}&=(\overline{v_Ha_H\alpha_H\sqrt{T_H}}+\overline{v_La_L\alpha_L\sqrt{T_L}})/(\overline{v_Ha_H\alpha_H}+\overline{v_La_L\alpha_L})\ .\end{aligned} \tag{11}$$

Here

$$\overline{f_H(t,\lambda)}=\frac{1}{\tau}\int_0^\tau f_H(t,\lambda)dt,\quad \overline{f_L(t,\lambda)}=\frac{1}{\tau}\int_0^\tau f_L(t,\lambda)dt\ .$$

When reservoir temperatures are constant, areas of contact a_H and a_L are constants, $v_H(t)=1$ for $0\le t\le t_H$, $v_H(t)=0$ for $t_H<t\le\tau$, $v_L(t)=0$ for $0\le t\le t_H$, and $v_L(t)=1$ for $t_H<t\le\tau$, it follows from (11) that

$$\tilde{\mathcal{P}}_{\max}(t_H)=\frac{a_H\alpha_Ha_L\alpha_Lt_H(\tau-t_H)/\tau}{a_H\alpha_Ht_H+a_L\alpha_L(\tau-t_H)}(\sqrt{T_H}-\sqrt{T_L})^2\ . \tag{12}$$

Maximizing (12) with t_H: $0<t_H<\tau$, we get the limiting value of the power

$$\hat{\mathcal{P}}_{\max}=\frac{a_H\alpha_Ha_L\alpha_L}{(\sqrt{a_H\alpha_H}+\sqrt{a_L\alpha_L})^2}(\sqrt{T_H}-\sqrt{T_L})^2\ ,$$

which coincides with the result of Curzon and Ahlborn.[1] The optimal $\hat{t}_H$ in (12) is $\hat{t}_H=\tau\sqrt{a_L\alpha_L}/(\sqrt{a_H\alpha_H}+\sqrt{a_L\alpha_L})$.

For Problem 2″, the distributed-parameter model,

$$\mathcal{P}_{\max}\le\max_T\left\{\frac{1}{\tau}\int_0^\tau\left[\int_{A_H}v_H\alpha_H(T_H-T)\left[1+\frac{\lambda}{T}\right]da+\int_{A_L}v_L\alpha_L(T_L-T)\left[1+\frac{\lambda}{T}\right]da\right]dt-\lambda\delta\right\}, \tag{13}$$

where $\lambda<0$ is a Lagrange multiplier, and $T(t,\xi)>0$ is a control. Here we break the constraint (9) in the Problem 2″ and get a range of estimates depending on λ. The term $\lambda\delta$ in (13) may be omitted, because the conditions $\lambda<0$ and

$\delta \le 0$ immediately imply $\lambda\delta \ge 0$. Maximizing the first term of the integrand with T we get the optimal argument $\hat{T}_1 = \sqrt{-\lambda T_H(t,\xi)}$ and the optimal value of this term

$$g_H(t,\xi,\lambda) = v_H(t)\alpha_H(\xi)[\sqrt{T_H(t,\xi)} - \sqrt{-\lambda}]^2 .$$

From the same maximization of the second term in (13) we get $\hat{T}_2 = \sqrt{-\lambda T_L(t,\xi)}$ and the optimal value of the term

$$g_L(t,\xi,\lambda) = v_L \alpha_L(\xi)[\sqrt{T_L(t,\xi)} - \sqrt{-\lambda}]^2 .$$

It is evident that

$$\mathcal{P}_{\max} \le \frac{1}{\tau}\int_0^\tau \left[\frac{1}{a_H(t)} \int_{A_H(t)} f_H(t,\xi,\lambda) da + \frac{1}{a_L(t)} \int_{A_L(t)} f_L(t,\xi,\lambda) da \right] dt ,$$

where $f_H = a_H g_H$ and $f_L = a_L g_L$. Minimizing the right part of this inequality with the variable $\lambda < 0$, we get the condition for the optimum $\hat{\lambda}$ and the limiting value of the power. The result formally coincides with expression (11), but the averaging is carried out over the surface area as well as over time. Here

$$\overline{f_H} = \frac{1}{\tau}\int_0^\tau \left[\frac{1}{a_H} \int_{A_H(t)} f_H(t,\xi,\lambda) da \right] dt, \quad \overline{f_L} = \frac{1}{\tau}\int_0^\tau \left[\frac{1}{a_L} \int_{A_L(t)} f_L(t,\xi,\lambda) da \right] dt .$$

For constant temperatures of reservoirs, it follows from (11) that

$$\tilde{\mathcal{P}}_{\max} = \frac{\gamma_H \gamma_L}{\gamma_H + \gamma_L} (\sqrt{T_H} - \sqrt{T_L})^2 , \tag{14}$$

where

$$\gamma_H = \frac{1}{\tau}\int_0^\tau v_H(t) \int_{A_H(t)} \alpha_H(\xi) d\xi, \quad \gamma_L = \frac{1}{\tau}\int_0^\tau v_L(t) \int_{A_L(t)} \alpha_L(\xi) d\xi .$$

When the reservoir temperatures are constant and the working fluid cannot contact two reservoirs simultaneously, that is, $v_H(t)=1$ for $0 \le t \le t_H$, $v_H(t)=0$ for $t_H < t \le \tau$, $v_L(t)=0$ for $0 \le t \le t_H$, and $v_L(t)=1$ for $t_H < t \le \tau$, it follows from (14) that

$$\tilde{\mathcal{P}}_{\max}(t_H) = \frac{\beta_H \beta_L t_H(\tau - t_H)/\tau}{\beta_H t_H + \beta_L(\tau - t_H)} (\sqrt{T_H} - \sqrt{T_L})^2 , \tag{15}$$

where

$$\beta_H = \frac{1}{t_H}\int_0^{t_H} \int_{A_H(t)} \alpha_H(\xi) d\xi, \quad \beta_L = \frac{1}{\tau - t_H}\int_{t_H}^\tau \int_{A_L(t)} \alpha_L(\xi) d\xi .$$

Maximizing (15) for fixed β_H and β_L with t_H: $0 < t_H < \tau$, we get the limiting value of the power

$$\hat{\mathcal{P}}_{\max} = \frac{\beta_H \beta_L}{(\sqrt{\beta_H} + \sqrt{\beta_L})^2} (\sqrt{T_H} - \sqrt{T_L})^2 . \tag{16}$$

Estimate (16) is a direct generalization of the result[1] for the engine with constant contact surfaces and space-varying heat-transfer coefficients. For engines with varying contact surfaces it is better to use (15) than (16).

Let us give an example of calculating β_H and β_L for the working fluid in a cylinder like the previous example but with variable hot area. Assuming that the thickness of the walls does not vary in space, we have

$$\beta_H = \alpha_H^W 2\pi r \frac{1}{t_H}\int_0^{t_H} x(t) dt + \alpha_H^P \pi r^2$$

and

$$\beta_L = \alpha_L^B \pi r^2 ,$$

where α_H^W is a heat-transfer coefficient for the walls, α_H^P is a heat-transfer coefficient for the piston, and α_L^B is a heat-transfer coefficient for the cold end of the cylinder. Methods for obtaining these coefficients are studied in the theory of heat transfer.[27]

Expressions (11) and (14) for the distributed-parameter model may be used to estimate the power output of real Stirling cycle engines. For such engines we assume that $v_H(t) = v_L(t) = 1$ for $0 \le t \le \tau$. Information needed for such analysis is the shape of contact surfaces $A_H(t)$ and $A_L(t)$, the heat-transfer coefficients $\alpha_H(\xi)$ and $\alpha_L(\xi)$, and the temperatures of reservoirs $T_H(t,\xi)$ and $T_L(t,\xi)$.

This method can also lead to limits of efficiency with given average power output and to limits based on other criteria of performance. For example, the efficiency of a system of fixed mean power may be estimated. This problem has an analytic solution for constant-T reservoirs but must be solved by computation for the variable-T case. These results will be presented in a computationally oriented extension of this work.

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[1]F. L. Curzon and B. Ahlborn, Am. J. Phys. **43**, 22 (1975).
[2]B. Andresen, R. S. Berry, A. Nitzan, and P. Salamon, Phys. Rev. A **15**, 2086 (1977).
[3]P. Salamon and A. Nitzan, J. Chem. Phys. **74**, 3546 (1981).
[4]M. J. Ondrechen, B. Andresen, M. Mozurkewich, and R. S. Berry, Am. J. Phys. **49**, 681 (1981).
[5]L. I. Rozonoer and A. M. Tsirlin, Automat. Remote Control **44**, 55 (1983).
[6]V. N. Orlov, Dokl. Akad. Nauk SSSR **282**, 1352 (1985) [Sov. Phys. Dokl. **30**, 506 (1985)].
[7]V. N. Orlov and A. V. Rudenko, Automat. Remote Control **46**, 549 (1985).
[8]A. de Vos, Am. J. Phys. **53**, 570 (1985).
[9]R. D. Spence and M. J. Harrison, Am. J. Phys. **53**, 890 (1985).
[10]E. Rebham and B. Ahlborn, Am. J. Phys. **55**, 423 (1987).
[11]H. S. Leff, Am. J. Phys. **55**, 602 (1987).
[12]J. M. Gordon, Am. J. Phys. **57**, 1136 (1989).
[13]M. H. Rubin, Phys. Rev. A **22**, 1741 (1980).
[14]F. Fairén and J. Ross, J. Chem. Phys. **75**, 5485 (1981); **75**, 5490 (1981).
[15]P. Salamon, Y. B. Band, and O. Kafri, J. Appl. Phys. **53**, 197 (1982).
[16]J. Chen and Z. Yan, Phys. Rev. A **39**, 4140 (1989).
[17]K. H. Hoffmann, S. J. Watowich, and R. S. Berry, J. Appl. Phys. **58**, 2125 (1985).
[18]M. J. Ondrechen, B. Andresen, and R. S. Berry, J. Chem. Phys. **72**, 5118 (1984).
[19]O. Mullins and R. S. Berry, J. Chem. Phys. **88**, 723 (1984).
[20]G. R. Brown, S. Snow, B. Andresen, and P. Salamon, Phys. Rev. A **34**, 4370 (1986).
[21]S. J. Watowich, K. H. Hoffman, and R. S. Berry, J. Appl. Phys. **58**, 2893 (1985).
[22]S. J. Watowich and R. S. Berry, J. Chem. Phys. **90**, 4624 (1986).
[23]J. W. Warner and R. S. Berry, J. Phys. Chem. **91**, 2216 (1987).
[24]J. M. Gordon and Y. Zarmi, Am. J. Phys. **57**, 995 (1989).
[25]J. Serrin, *Mathematical Principles of Classical Fluid Mechanics* (Springer-Verlag, Berlin, 1957).
[26]H. B. Callen, *Thermodynamics* (Wiley, New York, 1960).
[27]A. J. Chapman, *Heat Transfer* (Macmillan, New York, 1960).