Fluctuations in the interface between two phases

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In continuation of an earlier paper [F. Schlögl and R. S. Berry, Phys. Rev. A 21, 2078 (1980)] (referred to as I), small fluctuations in the interface between two phases will be discussed and with this an error in I will be corrected. The dynamics of the fluctuations is connected with the quantum mechanics of the modified Pöschl-Teller potential. Moreover, the fluctuations in the solitary solution of a moving surface are studied and also brought into connection with a quantum-mechanical system. The regression spectrum of these fluctuations is discussed in dependence on the velocity of the interface. If this velocity exceeds a certain value, the character of the regression to the kink profile changes qualitatively.

I. INTRODUCTION

In an earlier paper of two of the authors,[1] hereinafter called I, the dynamics of small fluctuations in the interface between two coexistent phases was discussed. The system under consideration was a chemical model for a nonequilibrium phase transition of first order,[2] which is closely analogous to the gas-liquid system. This model will be discussed in a deterministic description in which the temporal behavior of a fluctuation is that of any perturbation, regardless of how it was initially prepared. (We shall consequently use the terms "fluctuation" and "perturbation" interchangeably in the present context.) As will be shown in what follows, the dynamics of these fluctuations can be identified mathematically with quantum mechanics of a particle in a so-called "modified Pöschl-Teller potential."[3] The treatment of this quantum-mechanical system makes the fluctuation dynamics more clear in some respects. Thus an error in I became apparent to the authors which will be corrected here. This can be done most easily in the treatment of the above-mentioned problems. Not only will the fluctuations in the steady kink be discussed but also fluctuations in solitary states. These solitary states describe a moving interface layer analogous to the evaporation or condensation of a gas-liquid system with a plane interface. In this case the quantum-mechanical analog is the motion in an unsymmetric potential with suitably modified boundary conditions for the admitted solutions.

During the writing of this paper the authors received a copy of a paper by Magyari[4] about the fluctuations in the kink. It is our aim to give additional results in the following.

II. SMALL FLUCTUATIONS IN THE STEADY INTERFACE LAYER

As shown in Ref. 2 a certain chemical reaction diffusion model leads to the dynamical equation

$$\dot{n} = \nabla^2 n + \varphi(n) \tag{2.1}$$

with

$$\varphi(n) = -n^3 + 3n^2 - \beta n + \gamma \tag{2.2}$$

and non-negative β,γ. If β is smaller than the critical value 3, there exists a range of positive γ such that Eq. (2.1) has three steady (i.e., time-independent) solutions $0 \leq n_1 \leq n_3 \leq n_2$ which are homogeneous in space. The solutions n_1, n_2 are stable and behave in many respects like the densities of the two gas-liquid phases. As the steady states are not states of thermodynamic equilibrium, we speak of a nonequilibrium phase transition. It is easily seen that in adequate units Eq. (2.1) with non-negative β,γ is the most general form of a cubic reaction diffusion equation for a positive definite physical quantity which exhibits bistability of two homogeneous steady states. Moreover, the dynamical equation (2.1) is valid, at least in the neighborhood of the critical point, for various other systems with a phase transition of first order. Therefore, the following discussion is more generally applicable than with respect to the chemical model only.

In this Section and in Sec. III we discuss the coexistence case only in which, as shown in Ref. 2, the unstable state $n_3=1$ is the arithmetic average of the stable states n_1,n_2. There also was shown explicitly that by introducing

$$\nu=n-1 \tag{2.3}$$

and

$$\nu_0=(n_2-n_1)/2\ , \tag{2.4}$$

the dynamical equation takes on the standard form

$$\dot{\nu}=\nabla^2\nu-\nu(\nu^2-\nu_0^2)\ . \tag{2.5}$$

The steady kink solution belonging to a plane interface layer perpendicular to the x_3 axis and separating the two homogeneous phases $\pm\nu_0$ is given by

$$\nu^*=\nu_0\tanh(\sigma x_3)=\nu_0\zeta\ , \tag{2.6}$$

where

$$2\sigma^2=\nu_0^2\ . \tag{2.7}$$

After the change from the variable x_3 to ζ the linearized differential equation for small deviations

$$\nu-\nu^*=\nu_0\psi \tag{2.8}$$

assumes the form

$$\dot{\psi}=\sigma^2(1-\zeta^2)\mathscr{D}_2^2\psi+\Delta\psi \tag{2.9}$$

with

$$\Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\ , \tag{2.10}$$

$$\mathscr{D}_l^\mu=\frac{\partial}{\partial\zeta}\left[(1-\zeta^2)\frac{\partial}{\partial\zeta}\right]+[l(l+1)-(1-\zeta^2)^{-1}\mu^2]\ . \tag{2.11}$$

Application of $\mathscr{D}_l^m$ with integer l,m to the Legendre function $P_l^m(\zeta)$ gives zero.

In paper I the differential equation (2.9) for ψ was transformed to new variables. The transformation used there, however, is not correct if the new variables are treated as independent, since the derivative with respect to time then has a different meaning before and after the transformation, dependent on which variables are held fixed. In the following this transformation will not be used.

Now let $X(x_1,x_2)$ be the solution of the two-dimensional boundary-value problem

$$(\Delta+k^2)X=0 \tag{2.12}$$

leading to eigenvalues k. If, for instance, the boundary conditions require vanishing X at infinity, there exists a complete set of solutions

$$X=J_n(kr)e^{in\phi} \tag{2.13}$$

with Bessel functions J_n of integer order n and with

$$x_1+ix_2=re^{i\phi}\ . \tag{2.14}$$

For any kind of boundary conditions in x_1,x_2 the separation ansatz

$$\psi=u(\zeta)X(x_1,x_2)e^{-\lambda t} \tag{2.15}$$

leads to

$$\mathscr{D}_2^\mu u=0 \tag{2.16}$$

with

$$\lambda=k^2+(4-\mu^2)\sigma^2\ . \tag{2.17}$$

The solutions $u(\zeta)$ of Eq. (2.16) are dependent on μ only, whereas the damping coefficient λ of the fluctuation mode, Eq. (2.15), depends on μ and k. Since, for a fixed value μ, the value k gives only an additive term in λ, we introduce

$$\bar{\lambda}=\lambda-k^2\ . \tag{2.18}$$

The special modes with vanishing k have space dependence with respect to x_3 only. Moreover, they are the least damped and thus constitute the most interesting fluctuations.

As Eq. (2.17) shows, $\bar{\lambda}$ is proportional to σ^2 which is a measure of the distance from the critical point. It is

$$2\sigma^2=3-\beta\ , \tag{2.19}$$

where 3 is the critical value of β. Thus the approach to the critical point is connected with a decrease of $\bar{\lambda}$ to zero, that means, with a "critical slowing down."

We can distinguish two classes of solutions $u(\zeta)$ of Eq. (2.16) belonging to our problem. The first one is the class of solutions u which vanish for infinite x_3, i.e., for $\zeta=\pm1$. Only two modes belong to this class, the Legendre functions

$$u=P_2^\mu(\zeta) \tag{2.20}$$

with μ equal to 2 and to 1. The first is

$$u_G=P_2^2(\zeta)=3(1-\zeta^2)=\frac{3}{\sigma}\frac{\partial\zeta}{\partial x_3}\ . \tag{2.21}$$

A small fluctuation of this type changes the steady kink solution $\nu^*(x_3)$ into

$$\nu(x_3)=\left[1+\epsilon\frac{\partial}{\partial x_3)}\right]\nu^*(x_3)=\nu^*(x_3+\epsilon)\ . \tag{2.22}$$

This is nothing but a shift of the kink in the direction. For this mode $\bar{\lambda}$ clearly vanishes because the shifted kink is steady as well. This mode is the so-called "Goldstone mode." Its existence is a consequence of the invariance of the dynamics with respect to a shift in x_3.

The only time-dependent mode which vanishes at infinity belongs to $\mu=1$,

$$u_L=P_2^1(\zeta)=-3\zeta(1-\zeta^2)^{1/2}$$
$$=-3\frac{\sinh(\sigma x_3)}{[\cosh(\sigma x_3)]^2} . \quad (2.23)$$

It is the first nontrivial perturbation mode and is of particular interest because it describes the long-time regression of a perturbation to the original density profile.

Whereas the two values $\bar{\lambda}$ of this first class are discrete, the second class of modes has a continuous spectrum of $\bar{\lambda}$. The second class is characterized by the asymptotic behavior for $x_3 \to +\infty$,

$$u_c \to \cos(qx_3+\delta) . \quad (2.24)$$

The singularity at $\zeta=1$ in the differential equation (2.16) leads to the existence of two independent solutions with behavior differing by a circulation of ζ in the complex plane around this singularity. By convention[5] the solution which behaves for $\zeta \to 1$ in the following way:

$$u(1+(\zeta-1)e^{i2\pi})=e^{-i\pi\mu}u(\zeta) \quad (2.25)$$

is called

$$u=\mathscr{P}_2(\zeta) . \quad (2.26)$$

By Eq. (2.6) we obtain for the plane wave

$$e^{iqx_3}=\left[\frac{1+\zeta}{1-\zeta}\right]^{iq/2\sigma} . \quad (2.27)$$

Thus the solution u, which for positive infinite x_3 becomes proportional to this plane wave, is proportional to

$$u=\mathscr{P}_2^{iq/\sigma}(\zeta) . \quad (2.28)$$

The solution which has the asymptotic behavior equation (2.24) is

$$u=\tfrac{1}{2}e^{i\delta}\mathscr{P}_2^{iq/\sigma}(\zeta)+\tfrac{1}{2}e^{-i\delta}\mathscr{P}_2^{-iq/\sigma}(\zeta) . \quad (2.29)$$

These solutions belong to the continuous spectrum of values

$$\bar{\lambda}=4\sigma^2+q^2 . \quad (2.30)$$

The lower limit of this spectrum is $4\sigma^2$ and belongs to

$$u_c=P_2(\zeta)=\tfrac{3}{2}\zeta^2-\tfrac{1}{2}$$
$$=1-\tfrac{3}{2}[\cosh(\sigma x_3)]^{-2} . \quad (2.31)$$

Summarizing, we can state that apart from the trivial Goldstone mode u_G there exists one discrete mode u_L with the reduced damping factor $\bar{\lambda}=3\sigma^2$ and a continuum of shorter-living modes u_c with $\bar{\lambda}$ greater than $4\sigma^2$. All modes show the critical slowing down. The mode u_L is of particular interest because it describes the long-time regression of perturbations onto the original density profile.

III. MODIFIED PÖSCHL-TELLER POTENTIAL

The one-dimensional potential

$$V(x)=-V_0[\cosh(\sigma x)]^{-2} \quad (3.1)$$

is called the modified Pöschl-Teller potential and is used as a model potential[3] in the Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)-E\right]u=0 \quad (3.2)$$

of a mass point.

The differential equation (2.16) written in the variable x_3 is

$$\left[\frac{\partial^2}{\partial x_3^2}+6\sigma^2[\cosh(\sigma x_3)]^{-2}-4\sigma^2+\bar{\lambda}\right]u=0 \quad (3.3)$$

and thus equivalent to Eq. (3.2) if we put

$$V_0=\frac{\hbar^2}{m}3\sigma^2 , \quad (3.4)$$

$$E=\frac{\hbar^2}{2m}(\bar{\lambda}-4\sigma^2) . \quad (3.5)$$

The different modes discussed in the preceding section correspond to the following motions of the mass point. There exist only two bound states with negative energy value E. The lowest level

$$E_0=-2\hbar^2\sigma^2/m , \quad (3.6)$$

the ground state, corresponds to the Goldstone mode u_G. The excited bound state with

$$E_1=-\hbar^2\sigma^2/(2m) \quad (3.7)$$

represents the long-living mode u_L. These modes are vanishing for infinite $|x_3|$. Then we have a continuous spectrum of values $E \geq 0$ belonging to scattering states u_c. For infinite x_3 they become harmonic waves of the form of Eq. (2.24) with

$$E=\hbar^2q^2/(2m) . \quad (3.8)$$

The wave number q in the x_3 direction enters, due to Eqs. (2.17) and (2.29), into the damping factor λ in exactly the same way as the wave number k of a plane wave perpendicular to the x_3 axis.

As a special case of the bound solutions of the problem which will be discussed in Sec. V the solutions of the Schrödinger equation (3.3) can be expressed by special hypergeometric functions, so the connection of the functions of Eq. (2.26) with certain hypergeometric functions can be traced.

IV. FLUCTUATIONS IN SOLITARY STATES

The static interface layer between two phases n_1,n_2 exists only if n_3 is equal to the arithmetic mean value $\bar{n}$ of n_1,n_2. If however it differs from the latter

$$n_3=\bar{n}+\nu_0 a \tag{4.1}$$

there exists, as shown in detail in paper I, a "solitary solution" of the dynamical equation

$$\dot{\nu}=\nabla^2\nu-(\nu-\nu_0 a)(\nu^2-\nu_0^2) \tag{4.2}$$

for the deviation ν of n from $\bar{n}$. This solution

$$\tilde{\nu}=\nu_0\zeta=\nu_0\tanh(\sigma s)\ , \tag{4.3}$$

$$s=x_3-ct\ , \tag{4.4}$$

describes an interface layer moving with constant velocity

$$c=2\sigma a \tag{4.5}$$

in the direction of the x_3 axis like a fluid surface if condensation or evaporation takes place.

If we insert the definition equations (2.4) and (2.7) of σ into Eq. (4.5), we easily obtain

$$c\sim(n_2-n_1)^{-1}[\varphi'(n_2)-\varphi'(n_1)] \tag{4.6}$$

which admits a simple interpretation. $\varphi'(n_i)$ are the reciprocal lifetimes of small homogeneous perturbations of the homogeneous steady states n_i and Eq. (4.6) states that c is greater, the larger the difference between these reciprocal lifetimes, which is evidently reasonable. For fixed value of this difference, c is smaller, the larger the difference of n_2 and n_1, which is also clear because the conversion time of n_1 into n_2 (or vice versa) will increase with increasing distance of n_1 and n_2.

The dynamical equation

$$\dot{\psi}=\nu_0^2(1-3\zeta^2+2a\zeta)\psi+\nabla^2\psi \tag{4.7}$$

for small fluctuations $\nu_0\psi$ from $\tilde{\nu}$ was transformed in paper I from the variable x_3 to ζ. For the same reason as given in Sec. II this transformation was not correct.

We shall give here a correct transformation with the additional slight change that s will be used as a new variable instead of ζ because it serves better to make the analogy to a quantum-mechanical system. We have to distinguish between the time derivatives for fixed x_3 or for fixed s

$$\left.\frac{\partial}{\partial t}\right|_{x_3}=\left.\frac{\partial}{\partial t}\right|_{s}-c\left.\frac{\partial}{\partial s}\right|_{t} \tag{4.8}$$

and obtain

$$\dot{\psi}=-L\psi+\Delta\psi \tag{4.9}$$

with the linear operator

$$L=-\frac{\partial^2}{\partial s^2}-2\sigma a\frac{\partial}{\partial s}+2\sigma^2\{2-3[\cosh(\sigma s)]^{-2}-2a\tanh(\sigma s)\}\ . \tag{4.10}$$

This is the equation of motion of small fluctuations in the inertial system of the kink which describes the moving interface layer.

In this inertial system there exist separable modes

$$\psi=e^{-\eta t}u(s)\chi(x_1,x_2) \tag{4.11}$$

with χ fulfilling Eq. (2.12). Thus we obtain

$$Lu=\bar{\lambda}u\ , \tag{4.12}$$

where $\bar{\lambda}$ again is defined by Eq. (2.18).

If we write

$$L=L_0+aL_1 \tag{4.13}$$

with

$$L_1=-2\sigma\frac{\partial}{\partial s}-4\sigma^2\tanh(\sigma s)\ , \tag{4.14}$$

the Goldstone mode

$$u_G\sim\frac{\partial\zeta}{\partial s}=\sigma(1-\zeta^2)\ , \tag{4.15}$$

which is a small shift in the inertial system of the kink, leading from $\tilde{\nu}(s)$ to

$$\tilde{\nu}(s+\epsilon)=\tilde{\nu}+\epsilon\frac{\partial\tilde{\nu}}{\partial s}\ , \tag{4.16}$$

fulfills

$$L_1u_G=0 \tag{4.17}$$

in addition to

$$L_0u_G=0\ . \tag{4.18}$$

This again reflects the invariance of the original problem against time shift. The shape of the Goldstone mode is independent of the velocity c.

V. SCHRÖDINGER EQUATION OF FLUCTUATIONS IN SOLITARY SOLUTIONS

With ζ defined by Eq. (4.3) the differential equation (4.12) for u has the explicit form

$$\frac{d^2u}{ds^2}+2a\sigma\frac{du}{ds}+[2\sigma^2(1-3\zeta^2+2a\zeta)+\bar{\eta}]u=0\ , \tag{5.1}$$

which differs from a Schrödinger equation similar to Eq. (3.2) by the occurrence of the derivative of first order. This term can, however, be removed by a transformation[6] to Liouville's normal form

$$u=e^{-a\sigma s}\eta(s) \tag{5.2}$$

which leads to

$$\frac{d^2\eta}{ds^2}+[2\sigma^2(1-3\zeta^2+2a\zeta)-\sigma^2a^2+\bar{\lambda}]\eta=0\ . \tag{5.3}$$

This form now is equivalent to the Schrödinger equation (3.2) with

$$V(s)=-\frac{\hbar^2}{m}\sigma^2[3(1-\zeta^2)+2a\zeta] \tag{5.4}$$

$$=-\frac{\hbar^2}{m}\sigma^2\{3[\cosh(\sigma s)]^{-2}+2a\tanh(\sigma s)\}\ , \tag{5.5}$$

$$E=\frac{\hbar^2}{2m}[\bar{\lambda}-(4+a^2)\sigma^2]\ . \tag{5.6}$$

The potential $V(s)$ is unsymmetric in s not only for finite s. The values of V are different for $s=+\infty$ and $-\infty$,

$$V(\pm\infty)=\mp 2\frac{\hbar^2}{m}\sigma^2a\ . \tag{5.7}$$

The minimum is

$$V_{\min}=-3\frac{\hbar^2}{m}\sigma^2\left[1+\frac{a^2}{9}\right] \tag{5.8}$$

belonging to

$$\zeta=\tanh(\sigma s)=a/3\ . \tag{5.9}$$

As n_3 lies between n_1 and n_2 Eqs. (2.4) and (4.1) show that the parameter a varies between -1 and $+1$. From Eq. (5.3) we see that a change of sign of a is equivalent to a change of sign of s; that means a space inversion. Thus we need only consider positive a.

We can already obtain deep insight into the nature of the spectrum of the non-Hermitian operator L [see Eq. (4.12)] by carefully analyzing the Schrödinger potential $V(s)$ of Eq. (5.5) which is schematically depicted in Fig. 1.

The behavior of the Schrödinger function $\eta(s)$ in the asymptotic region $s\to\pm\infty$ is given by

$$\eta(s)\to\exp\left[\pm\left[\frac{2m}{\hbar^2}[V(\pm\infty)-E]\right]^{1/2}s\right]\ . \tag{5.10}$$

With the expressions (5.6) and (5.7) this yields

$$\eta(s)\to\exp\{\pm[(2\mp a)^2\sigma^2-\bar{\lambda}]^{1/2}s\}\ . \tag{5.11}$$

Equation (5.2) then gives us the asymptotic behavior of our perturbation u,

$$u(s)=A_\pm\exp[(-a+b_\pm)\sigma s]$$
$$+B_\pm\exp[(-a-b_\pm)\sigma s]\ , \tag{5.12}$$

where

$$b_\pm=[(2\mp a)^2-\bar{\lambda}/\sigma^2]^{1/2} \tag{5.13}$$

and where the upper and lower signs belong to $s\to\pm\infty$, respectively. Now we have to require that u vanishes for both $s\to+\infty$ and $-\infty$. As a is positive this means that B_- has to vanish. $A_\pm$ and B_+ then are determined up to a common factor. If A_+ is not zero by accident, the range of possible values $\bar{\lambda}$ is given by

$$\bar{\lambda}_1=4(1-a)\sigma^2<\bar{\lambda}<\bar{\lambda}_3$$
$$=4(1+a)\sigma^2\ . \tag{5.14}$$

It is easily seen from Eq. (5.12) that the part

$$\bar{\lambda}_2=(2-a)^2\sigma^2<\bar{\lambda}<\bar{\lambda}_3 \tag{5.15}$$

belongs to modes which constitute damped oscillations in space for $s\to+\infty$, whereas the other part of the spectrum belongs to monotonically damped modes.

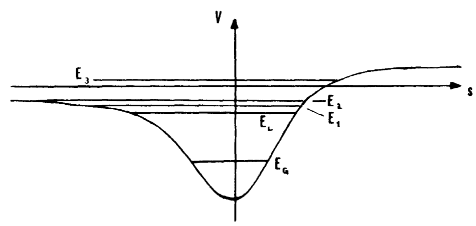

FIG. 1. Potential $V(s)$ and eigenvalues E of the Schrödinger equation belonging to the fluctuation regressions in the solitary solutions. E_G is Goldstone mode, E_L long-living discrete mode. Between E_1,E_2 lie the continuous modes with monotonic decrease in space, between E_2,E_3 those with damped oscillations in space.

It should be noted that (5.14) is not the only possibility to get nonexplosive solutions, (5.12). Another possibility is that $\bar{\lambda}$ is smaller than the upper limit of (5.14) and A_+ is zero by accident. This is just the case for the discrete modes of our problem which we discuss in the following. By a transformation

$$\zeta=\tanh(\sigma s)=2z-1\ , \tag{5.16}$$

Eq. (5.1) takes the form

$$z(1-z)\frac{d^2u}{dz^2}+(1+a-2z)\frac{du}{dz}+\left[6+\frac{\rho-a-1}{z}+\frac{\rho+a+1}{1-z}\right]u=0\ , \tag{5.17}$$

where

$$\rho=\bar{\lambda}/(4\sigma^2)\ . \tag{5.18}$$

In the ansatz

$$u=(1-z)^p z^q g(z)\ , \tag{5.19}$$

$g(z)$ becomes a power series uniformly convergent at $z=0$ and $z=1$, that is for $s=\pm\infty$, if

$$p^2-ap+a-1+\rho=0\ , \tag{5.20}$$

$$q^2+aq-a-1+\rho=0\ . \tag{5.21}$$

These equations are connected to each other by a change of sign in a. The differential equation for g then is

$$z(1-z)g''+[\gamma-(1+\alpha+\beta)z]g'-\alpha\beta g=0 \tag{5.22}$$

with

$$\alpha=p+q-2\ , \tag{5.23}$$

$$\beta=p+q+3\ , \tag{5.24}$$

$$\gamma=1+a+2q\ . \tag{5.25}$$

The polynomial solutions $g(z)$ of Eq. (5.22) are the hypergeometric functions

$$g=F(\alpha,\beta,\gamma;z) \tag{5.26}$$

with α or β an integer negative number including zero if, moreover, p,q both are positive definite. These conditions allow only two cases which give two bound states. The first, belonging to vanishing α and thus due to (5.18), (5.20), (5.21), and (5.23) to the values

$$p+q=2\ , \tag{5.27}$$

$$\bar{\lambda}_G=0\ , \tag{5.28}$$

is the Goldstone mode u_G. The second case belongs to the value -1 of α and represents the bound state u_L with

$$p=\tfrac{1}{2}(1-a)\ , \tag{5.29}$$

$$q=\tfrac{1}{2}(1+a)\ , \tag{5.30}$$

$$\bar{\lambda}_L=3\sigma^2(1-a^2)\ . \tag{5.31}$$

The corresponding solution g_L of Eq. (5.22) can be expressed as a polynomial of first order by a recursion formula for the coefficients

$$g_L(z)=1-2z/(1+a)\ . \tag{5.32}$$

That gives

$$u_L=(1-\zeta)^{1/2}\left[\frac{1+\zeta}{1-\zeta}\right]^{a/2}(a-\zeta) \tag{5.33}$$

$$=e^{a\sigma s}[a-\tanh(\sigma s)]/\cosh(\sigma s)\ . \tag{5.34}$$

A remarkable feature of the spectrum of our operator L can be seen in Fig. 2. At $|a|=\frac{1}{3}$ the eigenvalue $\bar{\lambda}_L$ crosses the lower boundary of the continuous spectrum $\bar{\lambda}_1$. This means that u_L for $|a|>\frac{1}{3}$ no longer describes the long-time regression of the fluctuations onto the profile of the solitary solution. As there is no longer a separation of time scales, the regression of perturbations can be expected to be quite complex.

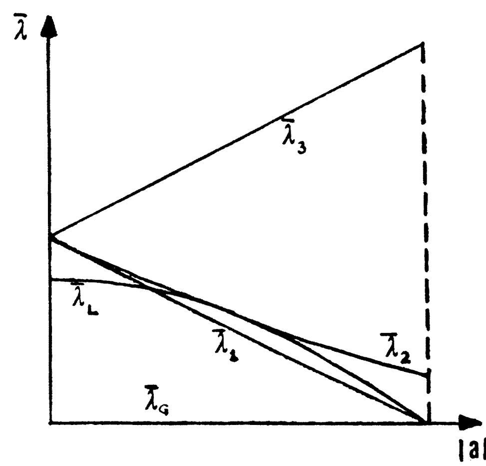

FIG. 2. Spectrum of the reduced damping factor of the fluctuation modes in the solitary solution. $\bar{\lambda}_G,\bar{\lambda}_L$ are the discrete modes. Between $\bar{\lambda}_1,\bar{\lambda}_2$ lie the continuous modes without oscillations, between $\bar{\lambda}_2,\bar{\lambda}_3$ those with damped oscillations in space.

VI. COMPARISON WITH PAPER I

The dynamical equation of the fluctuations in the steady kink (2.9) is identical with Eq. (3.1) of paper I. As carried out in paper I, the transformation of this equation to the new independent variables ξ_i was not correct. Here the transformation is avoided altogether. The results differ in the following: The fluctuation modes correctly derived are separable in time and space, unlike those given in I. Moreover, the spectrum of the modes has a continuous part.

The fluctuations in the solitary kink solution fulfill Eq. (4.7) which is identical to (4.12) of paper I. The transformation in paper I from x_3 to ζ was not correct and is here replaced by the transformation from x_3 to s. Unlike paper I, this paper is not restricted to small values of a, that is, to small velocities of the solitary kink. The development of the theory for finite a brings in a new set of nontrivial results.

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