Group representations in the Liouville representation and the algebraic approach

Michael P. Strand [a)]
Department of Chemistry, The University of Chicago, Chicago, Illinois 60637
Joint Institute for Laboratory Astrophysics, University of Colorado and National Bureau of Standards, Boulder, Colorado 80309

R. Stephen Berry
Department of Chemistry and The James Franck Institute, The University of Chicago, Chicago, Illinois 60637

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Group representations on the Liouville representation spaces are considered. It is shown that the state space $\mathscr{T}_1(H)$ of trace class operators on Hilbert space H and the observable space $\mathscr{L}(H)$ of bounded operators are completely reducible under physically induced representations of compact Hausdorff groups when appropriate topologies are used. For state space $\mathscr{T}_1(H)$ both the norm topology and the weak topology lead to complete reducibility, while for observable space $\mathscr{L}(H)$ the weak-$*$ topology—but not the norm topology—suffices. This leads to conservation laws, selection rules, and Wigner–Eckart theorems for the Liouville representation. It is shown that serious difficulties are encountered when a similar theory is attempted on the observable space and state space used in the algebraic approach.

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I. INTRODUCTION

Symmetry principles play a fundamental role in the understanding of physical laws and physical systems. Symmetry operations are conveniently represented in terms of group operations on objects representing the states and/or the observables of a system. When the states of a quantum mechanical system are represented by vectors in a Hilbert space H, most symmetry operations are represented by unitary operators, and a wealth of information may be obtained by applying the well-developed theory of unitary group representations. In this way one may derive conservation laws, selection rules, and quantitative relationships among certain transition amplitudes.

More recently, alternative phenomenologically oriented formulations of quantum mechanics have proven useful. They have the advantages of being formulated in terms of operationally defined quantities and of treating pure and mixed states on the same basis. One of these, the Liouville representation, is a straightforward generalization of the standard Hilbert space formulation in that it represents pure or mixed states as density operators on a fundamental Hilbert space H. Another, the algebraic approach, is defined more abstractly and has the property that it includes many more states than the Liouville representation. Both of these formulations are vector space theories, in which the vector spaces representing the states and observables are Banach spaces. Accordingly, while symmetry operations can be represented by operators acting on the space of states or the space of observables, they cannot be represented by unitary operators acting on these spaces; the theory of unitary group representations cannot be directly applied. Nevertheless, it is reasonable to expect that symmetry arguments will provide physical information similar to that which they provide in the pure state (i.e., Hilbert space) representation. Indeed, since the Banach space representations contain all information present in the Hilbert space representation, one might reasonably expect to obtain even stronger results. We show here that in fact there are conservation laws and selection rules for transitions which may be derived directly from group representations on the Liouville representation spaces which may only be obtained indirectly in the pure state-Hilbert space representation, where one applies group methods to the wavefunctions and then constructs averages over suitable mixed states.

In this paper we begin an investigation of representations of symmetry groups on the Banach spaces occurring in the Liouville representation and in the algebraic approach. We will restrict our attention to compact symmetry groups and deal primarily with the Liouville representation. We begin in Sec. II by recalling the fundamental results from the theory of unitary representations of compact groups on Hilbert spaces which are necessary to establish the desired physical results (e.g., conservation laws, selection rules, and Wigner–Eckart type theorems). We then show that the necessary mathematical results do not extend to the general continuous representation of a compact group on a Banach space. Thus, in order to draw the desired physical conclusions, it is necessary to establish these important results for the specific representations which are encountered in physical applications. In preparation for this, Sec. III reviews the Liouville space formulation of quantum theory, while Sec. IV is devoted to general aspects of group representations on topological vector spaces. In Sec. V we show that appropriate topologies make the state space and observable space of the Liouville representation completely reducible. This leads

[a)]Current address: Department of Physics, Texas A and M University, College Station, TX 77843.

to conservation laws, selection rules, and Wigner–Eckart type theorems for statistical properties describing quantum mechanical ensembles. These results may be obtained only indirectly (through explicit averaging procedures) from the application of group theory at the wavefunction (i.e., pure state) level. Finally in Sec. VI we show that difficulties are encountered when one attempts to generalize the unitary representation results to the representations occurring in the algebraic approach.

II. PROPERTIES OF UNITARY REPRESENTATIONS

Central to the theory and application of unitary group representations are the following classical results[1–3] (here G is a separable compact topological group, and all vector spaces are assumed to be complex):

(1) Let $U{:}G\times H\rightarrow H$ be a continuous unitary representation of G on Hilbert space H, and let H_1 be a subspace of H which is invariant under U [i.e., $U(g)H_1\subset H_1$ for all g in G]. Then there exists a closed subspace H_2 of H which is invariant under U and such that H is the direct sum $H=H_1\oplus H_2$ of H_1 and H_2. In fact, H_2 may be chosen to be the orthogonal complement of H_1 (i.e., $H_2=H_1^{\perp} =\{v\in H\,|\,\langle v|v_1\rangle=0 \text{ for all } v_1 \text{ in } H\}$).

(2) If $U{:}G\times H\rightarrow H$ is a continuous unitary representation of G on Hilbert space H which is irreducible (i.e., H has no nontrivial closed invariant subspaces), then H is finite-dimensional.

(3) (Schur's lemma): Let U_1: $G\times H_1\rightarrow H_1$ and U_2: $G\times H_2\rightarrow H_2$ be continuous unitary representations of G on Hilbert spaces H_1 and H_2, respectively. Let $\mathscr{S}: H_1\rightarrow H_2$ be a linear operator from H_1 to H_2 which commutes with the actions of the group [i.e., $\mathscr{S}U_1(g)=U_2(g)\mathscr{S}$ for all g in G; $\mathscr{S}$ is called an *intertwining operator* for U_1 and U_2]. Then, if U_1 and U_2 are irreducible, either $\mathscr{S}$ is the zero operator (i.e., $\mathscr{S}v_1=0$ for all v_1 in H_1) or $\mathscr{S}$ is an isomorphism (i.e., one-to-one and onto). In the latter case $\mathscr{S}$ is invertible and we have $U_1(g)=\mathscr{S}^{-1}U_2(g)\mathscr{S}$ and $U_2(g)=\mathscr{S}U_1(g)\mathscr{S}^{-1}$ so that the irreducible representations U_1 and U_2 are equivalent.

(4) (Peter–Weyl)[1–3]: Let $H=L^2(G)$ be the Hilbert space of square integrable functions (relative to Haar measure) on G, and let $U{:}G\times H\rightarrow H$ be the right regular representation of G defined by $[U(g)f](h)=f(hg)$, where g and h are in G. Then H may be decomposed into a direct sum of irreducible representations in which each unitary irreducible representation of G occurs with a (finite) multiplicity equal to its dimension. Thus, recalling that the dual object $\hat{G}$ of G is the collection of (equivalence classes of) irreducible unitary representations of G, we have

$$L^2(G)=\sum_{\hat{g}\in\hat{G}}\oplus\sum_{i=1}^{\dim(\hat{g})}\oplus H_{\hat{g}},$$

where $H_{\hat{g}}$ is a representation space for representation $\hat{g}$ and $\dim(\hat{g})$ is its dimension.

(5) (A. Gurevic[4]): If $U{:}G\times H\rightarrow H$ is a continuous unitary representation of G on Hilbert space H, then U is completely reducible. That is, for each $\hat{g}\in\hat{G}$ there is a cardinal number $m_{\hat{g}}$ such that $H=\Sigma_{\hat{g}\in\hat{G}}\oplus m_{\hat{g}}H_{\hat{g}}$, where $m_{\hat{g}}H_{\hat{g}}$ is the direct sum of $m_{\hat{g}}$ copies of $H_{\hat{g}}$. Each $m_{\hat{g}}H_{\hat{g}}$ is a primary subrepresentation of H, and, if $\hat{g}\neq\hat{g}'$, then $m_{\hat{g}}H_{\hat{g}}$ and $m_{\hat{g}'}H_{\hat{g}'}$ are disjoint. (Recall that two representations are *disjoint* if there exists no nonzero intertwining operator between them, and that a representation is *primary* if it cannot be decomposed into a direct sum of disjoint representations.)

(6) (Racah[5,6]): If G is a Lie group, there is a finite set of operators (the generalized Casimir invariants) whose eigenvalues uniquely characterize (to within equivalence) the irreducible representations of G. Thus the primary subrepresentations $m_{\hat{g}}H_{\hat{g}}$ of a unitary representation may be labeled by the eigenvalues of the generalized Casimir operators on the irreducible representation $\hat{g}$. Moreover the representations of the Casimir invariants are self-adjoint.

Conservation laws, selection rules, and Wigner–Eckart theorems may be easily derived from these results. By the theorem of Gurevic the Hilbert space of state vectors is reducible into a direct sum of primaries corresponding to each irreducible representation of G. Now a symmetry operation is by definition an invertible operation on a physical system which commutes with time development. That is, if one starts at time t_0 with two identical systems (A and B) and performs a symmetry operation on system A at time $t_A>t_0$ and the same symmetry operation on system B at time $t_B>t_0$, then at any time t greater than both t_A and t_B the two systems are again identical. Thus, if one has a group of symmetry operations the time translation operator $U(t_2,t_1)$ from t_1 to t_2 for any pair of times t_1 and t_2 commutes with the group of symmetry operations, i.e., $U(t_2,t_1)$ is an intertwining operator. Consequently, the transition amplitude $\langle\phi_{\hat{g}'}|U(t_2,t_1)|\psi_{\hat{g}}\rangle$ from any state $\psi_{\hat{g}}$ in primary $m_{\hat{g}}H_{\hat{g}}$ to any state $\phi_{\hat{g}'}$ in a different (i.e., disjoint) primary $m_{\hat{g}'}H_{\hat{g}'}$ is zero (by definition of disjoint representations). Because the Casimir operators are self-adjoint, they represent observables; we see that the values of these observables are conserved.

Selection rules and Wigner–Eckart theorems may be derived similarly. Suppose that one has a system made up of two subsystems (A and B). The Hilbert space H of pure states of the system is the tensor product $H=H_A\otimes H_B$ of the Hilbert spaces describing the subsystems. Each of these spaces has a direct sum decomposition into disjoint primaries; $H=\Sigma_{\hat{g}\in\hat{G}}m_{\hat{g}}H_{\hat{g}}$, $H_A=\Sigma_{\hat{g}\in\hat{G}}m_{\hat{g}}^AH_{\hat{g}}$, and $H_B=\Sigma_{\hat{g}\in\hat{G}}m_{\hat{g}}^BH_{\hat{g}}$. Suppose that at time t_0 subsystem A is described by vector $\phi_{\hat{g}_A}^A$ in primary $\hat{g}_A$ and subsystem B is described by vector $\phi_{\hat{g}_B}^B$ in primary $\hat{g}_B$. Then the transition amplitude $\langle\psi_{\hat{g}}|U(t_2,t_1)|\phi_{\hat{g}_A}^A\otimes\phi_{\hat{g}_B}^B\rangle$ into a state $\psi_{\hat{g}}$ of the entire system in primary $\hat{g}$ will be zero unless the irreducible representation $\hat{g}$ is contained in the product $\hat{g}_A\times\hat{g}_B$. This provides selection rules for changes in the eigenvalues of the generalized Casimir invariants. Furthermore, the Wigner–Eckart theorem is obtained from Schur's lemma which implies that any intertwining operator [such as the time translation operator $U(t_2,t_1)$] depends upon only one parameter for each irredicuble subspace in H and each pair of irreducible subspaces in H_A and H_B.

It is clear that if the six results listed above can be extended to representations on Banach spaces, then conservation laws, selection rules, and Wigner–Eckart theorems are valid for the Banach space representations of quantum theory. Accordingly, we will see which of these results generalize to isometric representations on Banach spaces. First of all, it is not ture that every closed invariant subspace B_1 of a continuous isometric representation $\mathscr{U}: G \times B \to B$ may be complemented by a closed invariant subspace B_2 such that $B = B_1 \oplus B_2$. In particular, let $B = \mathscr{L}(H)$ be the Banach space of bounded operators on Hilbert space H; let $B_1 = \text{Com}(H)$ be the Banach space of compact operators on H. Then, if $U: G \times H \to H$ is a continuous unitary representation of compact group G on H, $\mathscr{U}: G \times B \to B$ defined by $\mathscr{U}(g)\mathscr{O} = U(g)\mathscr{O}U^{-1}(g)$ is a continuous isometric representation of G on B which leaves B_1 invariant. But B_1 has no complementing subspace[7] in B so it certainly has no invariant complementing subspace. Since establishing the existence of complementary invariant subspaces is a key step in proving the complete reducibility of an arbitrary unitary representation, this failure suggests that not every continuous isometric representation on a Banach space will be completely reducible.

Secondly, while not every continuous irreducible representation of a compact group G on an arbitrary topological vector space is finite-dimensional, this statement is true if the topological vector space has a nonzero continuous linear functional.[3] The Hahn–Banach theorem thus implies that every continuous irreducible representation of G on a Banach space is finite-dimensional.

Thirdly, since by (2) every irreducible representation of G on a Banach space is finite-dimensional, Schur's lemma as stated applies to Banach space representations.

The Peter–Weyl theorem is specifically a Hilbert space result and need not be generalized.

It is not true that every continuous isometric representation $\mathscr{U}: G \times B \to B$ of a compact group is completely reducible. Specifically, the representation of the circle group $S^1 = \{e^{i\theta} | 0 \leqslant \theta < 2\pi\}$ on the space $B = L^1(S^1)$ of integrable functions on S^1 given by $[\mathscr{U}(\theta)f](\phi) = f(\theta + \phi)$ is not completely reducible.[8]

Finally, since the irreducible representations are all finite-dimensional, Racah's theorem on generalized Casimir invariants is unchanged.

Thus, in order to prove the conservation laws, selection rules, and Wigner–Eckart theorems it must be demonstrated that the particular representations encountered in physical applications are in fact completely reducible. We show in Sec. V that the group representations on the spaces of the Liouville representations are completely reducible.

III. THE LIOUVILLE REPRESENTATION

In the Liouville representation each state is represented by an operator ρ on Hilbert space H which satisfies:

DO1. ρ is self-adjoint, i.e., $\rho^\dagger = \rho$.
DO2. ρ is nonnegative, i.e., $\langle\psi|\rho|\psi\rangle \geqslant 0 \ \forall \ \psi \in H$.
DO3. ρ has unit trace, i.e., ρ is trace class and $\text{tr}\rho = 1$.

Any such operator is called a density operator (DO). State space is defined to be the complex vector space spanned by the density operators. It is precisely the Banach space of trace class operators on H and is denoted by $\mathscr{T}_1(H)$. State space thus has a norm (the trace class norm) given by

$$\|A\|_1 = \text{tr}[(A^+A)^{1/2}]. \tag{3.1}$$

Notice that each density operator has a state space norm of one.

Observables are represented by the bounded self-adjoint operators on H. Observable space is defined to be the complex vector space spanned by the observable operators. It is precisely the Banach space of all bounded operators on H and is denoted by $\mathscr{L}(H)$. Observable space thus has a norm (the operator norm) given by

$$\|\mathscr{O}\| = \sup_{\substack{\psi \in H \\ \langle\psi|\psi\rangle = 1}} \langle\mathscr{O}\psi|\mathscr{O}\psi\rangle^{1/2}. \tag{3.2}$$

The expectation value of observable $\mathscr{O}$ when the system is described by density operator ρ is given by $\text{tr}(\mathscr{O}\rho)$. This may be extended to a sesquilinear form on $\mathscr{L}(H) \times \mathscr{T}_1(H)$ by

$$(\mathscr{O}, A) \to \langle\langle\mathscr{O}|A\rangle\rangle \equiv \text{tr}(\mathscr{O}^+A). \tag{3.3}$$

$\mathscr{L}(H)$ is the dual of $\mathscr{T}_1(H)$,[9,10] where the duality is expressed by the sesquilinear form $\langle\langle \ | \ \rangle\rangle$. Furthermore, the norm of bounded operator $\mathscr{O}$ considered as a linear functional on $\mathscr{T}_1(H)$ coincides with its norm when it is considered as a bounded operator on H.[9,10] Thus

$$\|\mathscr{O}\| = \sup_{\substack{A \in \mathscr{T}_1(H) \\ \|A\|_1 = 1}} |\langle\langle\mathscr{O}|A\rangle\rangle|. \tag{3.4}$$

The norm of $A \in \mathscr{T}_1(H)$ is similarly given by[9,10]

$$\|A\|_1 = \sup_{\substack{\mathscr{O} \in \mathscr{L}(H) \\ \|\mathscr{O}\| = 1}} |\langle\langle\mathscr{O}|A\rangle\rangle|. \tag{3.5}$$

The norms $\|\cdot\|_1$ and $\|\cdot\|$ naturally define topologies on state space $\mathscr{T}_1$ and observable space $\mathscr{L}$. However, the duality between state space and observable space provides each with an alternative topology. The weak topology[9,11] on $\mathscr{T}_1$ is defined to be the weakest vector space topology such that for each $\mathscr{O} \in \mathscr{L}$ the expectation value function $E_{\mathscr{O}}^+: \mathscr{T}_1 \to C$ defined by $E_{\mathscr{O}}^+(A) = \langle\langle\mathscr{O}|A\rangle\rangle$ is continuous. It has the property that for any topological vector space X a linear function $f: X \to \mathscr{T}_1$ is continuous if and only if the composite function $E_{\mathscr{O}}^+ \circ f: X \to C$ is continuous for each $\mathscr{O} \in \mathscr{L}$. Similarly the weak-$*$ topology[9,11] on $\mathscr{L}$ is defined to be the weakest vector space topology such that for each $A \in \mathscr{T}_1$ the expectation value function $E_A: \mathscr{L}_1 \to \mathbb{C}$ defined by $E_A(\mathscr{O}) = \text{tr}\mathscr{O}^+A$ is continuous. It has the property that for any topological vector space X a linear function $f: X \to \mathscr{L}$ is continuous if and only if the composite function $E_A \circ f: X \to \mathbb{C}$ is continuous for each $A \in \mathscr{T}_1$.

State space $\mathscr{T}_1$ with either the norm or the weak topology and observable space with either the norm or weak-$*$ topology are locally convex and Hausdorff.[11]

IV. GROUP REPRESENTATIONS ON TOPOLOGICAL VECTOR SPACES

Throughout this section G is a compact Hausdorff topological group with Haar measure dg, V is a locally convex topological vector space, and $\Pi: G \to \mathscr{L}(V)$ is a group representation on V such that $(g, v) \mapsto \Pi(g)v$ from $G \times V$ to V is jointly continuous. We are most interested in the case where V does not have a Hilbert space structure. We begin by considering direct sum decompositions of V and then define complete reducibility of representation Π.

Definition: Suppose J is a finite index set and that for each $j\in J$ there exists a nonzero linear operator P_j on V such that

ADS 1. $P_iP_j = 0$ if $i \neq j$.
ADS 2. $P_iP_i = P_i$.
ADS 3. $\Sigma_{j\in J}P_j = I$ the identity operator on V.

Then we say we have an *algebraic direct sum* (*ADS*) *decomposition* of V and that V is *the algebraic direct sum* of $\{V_j\}_{j\in J}$ and we write $V = \Sigma_{j\in J} \oplus V_j$. Here $V_j = P_jV$ is the range of operator P_j and is an *algebraic direct summand* of V.

Definition: Suppose J is an index set and that for each $j\in J$ there exists a nonzero linear operator P_j on V such that

TDS 1. $P_iP_j = 0$ if $i \neq j$.
TDS 2. $P_iP_i = P_i$.
TDS 3. Each P_j is continuous.
TDS 4. $\Sigma_{j\in J}P_j = I$, where $I\in\mathscr{L}(V)$ is the identity operator on V and the sum converges in the strong operator topology on $\mathscr{L}(V)$ (i.e., for each $v\in V$ and each neighborhood N of v there exists a finite set $J'\subset J$ such that for each finite set $J''\subset J$ satisfying $J'\subset J''$ the finite sum $\Sigma_{j\in J''}P_jv$ is in neighborhood N).

Then we say we have a *topological direct sum* (*TDS*) *decomposition* of V and that V is a topological direct sum of $\{V_j\}_{j\in J}$ and we write $V = \Sigma_{j\in J} \oplus V_j$. Here $V_i = P_iV$ is the range of operator P_i and is a *topological direct summand* of V.

Definition: If $V = V_1 \oplus V_2$ is an *algebraic* (respectively, *topological*) *direct sum* then V_2 is called an *algebraic* (respectively, *topological*) *complement* of V_1, and vice versa. We also say that V_1 is *complemented* by V_2.

These definitions call for some comment:

(1) When V is finite-dimensional, there is no difference between an algebraic direct sum and a topological direct sum. This is because (a) the index set J is necessarily finite so that no topology is needed to define the sum $\Sigma_{j\in J}P_j$ and (b) on a finite-dimensional vector space every linear operator P_j is continuous.

(2) While every subspace V_1 of V is algebraically complemented by some subspace V_2, it need not have a topological complement even if it is closed. However, V_1 will have a topological complement if it is closed and satisfies any one of the following three conditions[3]:

(a) V_1 has finite codimension,

(b) V_1 has finite dimension (the assumption that V is locally convex is crucial here),

(c) V is linearly homeomorphic with a Hilbert space.

(3) The above definition of topological direct sums does not correspond precisely with the universally defined direct sum. It is, however, simply related to the universal topological direct sum and the universal topological direct product.[12,13]

Definition: Suppose J is an index set and V_j is a topological vector space for each $j\in J$. The *universal topological direct sum* of $\{V_j\}_{j\in J}$ is the (unique) locally convex topological vector space (denoted by $\tilde{\Sigma}_{j\in J} \oplus V_j$) equipped with a set of continuous linear maps I'_j: $V_j \rightarrow \tilde{\Sigma}_{j\in J} \oplus V_j$ such that the following universal mapping property is satisfied. Given any locally convex topological vector space W and any set of continuous linear maps F_j: $V_j \rightarrow W$, there exists a unique continuous linear map F: $\tilde{\Sigma}_{j\in J} \oplus V_j \rightarrow W$ such that $F_j = F\circ I'_j$.

Concretely, $\tilde{\Sigma}_{j\in J} \oplus V_j$ is the vector space of all functions on J such that $f(j)\in V_j$ and $f(j) = 0$ for all but a finite number of $j\in J$. The collection of subsets $\mathscr{O} = \{f\in\tilde{\Sigma}_{j\in J} \oplus V_j \,|\, f(j)\in\mathscr{O}_j,\ \mathscr{O}_j$ open in $V_j\}$ is a local base at 0 for the universal direct sum topology.

The insertion operators I_j: $V_j \rightarrow V = \Sigma_{j\in J} \oplus V_j$, which are restrictions of the identity operator I: $V \rightarrow V$ to the closed subspaces V_j are continuous and linear so that by the universal mapping property there exists a continuous linear function I:$\tilde{\Sigma}_{j\in J} \oplus V_j \rightarrow V$ such that $I_j = I\circ I'_j$. I is one-one so $\tilde{\Sigma}_{j\in J} \oplus V_j$ is isomorphic to a subspace of V. Furthermore the topology on V is weaker than the $\{I_j\}_{j\in J}$-final linear topology on V.

Definition: Suppose J is an index set and V_j is a topological vector space for each $j\in J$. The *universal topological direct product* of the V_j is the (unique) locally convex topological vector space $\tilde{\Pi}_{j\in J} V_j$ equipped with a set of continuous linear maps π_j:$\tilde{\Pi}_{j\in J} V_j \rightarrow V_j$ such that the following universal mapping property is satisfied. Given any locally convex topological vector space W and any set of continuous linear maps F_j :$W \rightarrow V_j$, there exists a unique continuous linear map F:$W \rightarrow \tilde{\Pi}_{j\in J} V_j$ such that $F_j = \pi_j\circ F$.

Concretely $\tilde{\Pi}_{j\in J} V_j$ is the vector space of all functions on J such that $f(j)\in V_j$. The collection of subsets of the form $\mathscr{O} = \pi_j^{-1}(\mathscr{O}_j)$ for $\mathscr{O}_j$ open in V_j is a subbase for the product topology.

The projection operators P_j: $V = \Sigma_{j\in J} \oplus V_j \rightarrow V_j$ are continuous and linear so by the universal mapping property there exists a continuous linear map P: $V \rightarrow \tilde{\Pi}_{j\in J} V_j$ such that $P_j = \pi_j\circ P$. Because no $v\in V$ is annihilated by all P_j, P is one-one and V is isomorphic to a subspace of $\tilde{\Pi}_{j\in J} V_j$. The topology on V must be stronger than the subspace topology inherited from $\tilde{\Pi}_{j\in J} V_j$.

The relationship between our definition of a topological direct sum and the universal topological direct sum and universal topological direct product may be summarized as follows: For our purposes a topological direct sum of locally convex spaces $\{V_j\}_{j\in J}$ is any locally convex topological vector space V which is (linearly isomorphic to) a subspace of the universal direct product $\tilde{\Pi}_{j\in J} V_j$ and which contains the universal direct sum $\tilde{\Sigma}_{j\in J} \oplus V_j$ as a vector subspace such that the topology on V is stronger than the subspace topology inherited from $\tilde{\Pi}_{j\in J} V_j$ and such that it induces a topology on the universal direct sum $\tilde{\Sigma}_{j\in J} \oplus V_j$ which is weaker than the universal direct sum topology.

(4) When the index set J is finite, the universal direct sum and the universal direct product of $\{V_j\}_{j\in J}$ are equal so that any two direct sums of $\{V_j\}_{j\in J}$ are isomorphic.[13] Hence the topological direct sum agrees with the universal topological direct sum.

(5) When the index set J is infinite, the universal direct product is much bigger than the universal direct sum. Consequently, two different vector spaces V and W may be direct sums of the same $\{V_j\}_{j\in J}$. V and W may differ both in terms

of their topologies and in terms of their sets. This nonuniqueness is easily illustrated by taking direct sums of an infinite number of Banach spaces B_j. For all positive p, let B^p be the set of functions f such that $f(j)\in B_j$ and $\|f\|_p = (\Sigma_{j\in J}\|f(j)\|^p)^{1/p}$ is finite. Here $\|f(j)\|$ is the norm of $f(j)\in B_j$. This gives a topological direct sum for each $p>0$. While $p=1$ is sometimes taken as the definition of the Banach space direct sum,[1] there is nothing canonical about this choice.

We conclude this section with three definitions.

Definition: A closed subspace V_1 of V is *invariant* under the representation $\Pi: G\rightarrow\mathscr{L}(V)$ if $\pi(g)v_1\in V_1$ for all $v_1\in V_1$ and all $g\in G$.

Definition: The representation $\Pi: G\rightarrow\mathscr{L}(V)$ is *irreducible* if V has no nontrivial closed invariant subspaces.

Definition: The representation $\Pi: G\rightarrow\mathscr{L}(V)$ is *completely reducible* if V has a direct sum decomposition into invariant irreducible subspaces.

V. THE COMPLETE REDUCIBILITY OF $\mathscr{I}_1(H)$ AND $\mathscr{L}(H)$

In this section we prove the main results of this paper: The projection operators from state space (observable space) to the standard irreducible tensorial operators decompose state space (observable space) into a direct sum of invariant subspaces when an appropriate topology is used. We begin by considering the properties of the physically induced representations on state space $\mathscr{I}_1(H)$ and observable space $\mathscr{L}(H)$.

Let $T: G\times H\rightarrow H$ be a jointly continuous unitary representation of a compact, Hausdorff group G on Hilbert space H, where H is assumed to have the norm topology. This induces several representation of G on state space $\mathscr{I}_1(H)$. Define, for $A\in\mathscr{I}_1(H)$,

$$\mathscr{T}_l(g)A \equiv T(g)A, \tag{5.1}$$

$$\mathscr{T}_r(g)A \equiv AT^+(g), \tag{5.2}$$

$$\mathscr{T}_a(g)A \equiv T(g)AT^+(g). \tag{5.3}$$

These are the left, right, and adjoint representations of G, respectively. $\mathscr{T}_l$ and $\mathscr{T}_r$ may be combined to form a representation $\mathscr{T}_p$ of the product group $G\times G$ by defining

$$\mathscr{T}_p(g,h)A = T(g)AT^+(h) = \mathscr{T}_l(g)\mathscr{T}_r(h)A. \tag{5.4}$$

The adjoint representation is the product representation restricted to the diagonal: $\mathscr{T}_a(g) = \mathscr{T}_p(g,g)$. Because of the unitarity of T, all of these are isometric representations, i.e.,

$$\|\mathscr{T}A\|_1 = \|A\|_1 \quad \forall A\in\mathscr{I}_1(H). \tag{5.5}$$

For each of the representations $\mathscr{I}_1$, $\mathscr{T}_r$, and $\mathscr{T}_a$ the mapping $(g,A)\mapsto\mathscr{T}(g)A$ is jointly continuous from $G\times\mathscr{I}_1$ to $\mathscr{I}_1$ when $\mathscr{I}_1$ is given either the norm or the weak topology. Similarly the mapping $(g,h,A)\rightarrow\mathscr{T}_p(g,h)A$ is jointly continuous from $G\times G\times\mathscr{I}_1$ to $\mathscr{I}_1$. To prove these assertions, assume first that $g\mapsto\mathscr{T}(g)A$ from G to $\mathscr{I}_1$ is norm continuous for each $A\in\mathscr{I}_1$. Then

$$\begin{aligned}&\|\mathscr{T}(g)A - \mathscr{T}(g_0)A_0\|_1\\ &\quad\leqslant\|\mathscr{T}(g)(A-A_0)\|_1 + \|[\mathscr{T}(g)-\mathscr{T}(g_0)]A_0\|_1\\ &\quad= \|A-A_0\|_1 + \|[\mathscr{T}(g)-\mathscr{T}(g_0)]A_0\|_1,\end{aligned}$$

which goes to zero as $A\rightarrow A_0$ and $g\rightarrow g_0$ so joint continuity has been established. Thus, since if $g\mapsto\mathscr{T}(g)A$ is norm continuous it is also weakly continuous, it is only necessary to show that it is norm continuous. We first prove this for the left representation. Choose $\epsilon>0$. Let $A = \Sigma_{j\in J}|\phi_i\rangle\lambda_i\langle\psi_i|$ be the polar representation[10] of A. Thus $\langle\phi_i|\phi_j\rangle = \delta_{ij}$, $\langle\psi_i|\psi_j\rangle = \delta_{ij}$, $\lambda_i\geqslant 0$, $\Sigma_{i\in I}\lambda_i = \|A\|_1$ and I is the set of integers. Choose $N\in I$ such that $\Sigma_{i=N+1}^{\infty}\lambda_i < \epsilon/4$ and choose neighborhood $\mathscr{U}$ of g_0 such that $\|[T(g)-T(g_0)]\phi_i\| < \epsilon/2N\lambda_i$ for $i=1,\dots,N$. Then, defining $A_N = \Sigma_{i=1}^{N}|\phi_i\rangle\lambda_i\langle\psi_i|$ and $A' = A - A_N$, we have

$$\begin{aligned}&\|[\mathscr{T}_l(g)-\mathscr{T}_l(g_0)]A\|_1\\ &\quad\leqslant\|[T(g)-T(g_0)]A_N\|_1 + \|[T(g)-T(g_0)]A'\|_1\\ &\quad\leqslant\sum_{i=1}^{N}\lambda_i\|[T(g)-T(g_0)]\phi_i\| + \|T(g)-T(g_0)\|\cdot\|A'\|_1\\ &\quad<\epsilon/2 + 2(\epsilon/4) = \epsilon.\end{aligned}$$

A similar argument proves that the right representation $\mathscr{T}_r$ is norm continuous. It is now easy to show that the product representation is continuous, i.e., that $(g,h)\mapsto\mathscr{T}_p(g,h)A$ is continuous for each $A\in\mathscr{I}_1$. Fix $\epsilon>0$. Choose a neighborhood $\mathscr{U}_1$ of g_0 such that $\|[\mathscr{T}_l(g)-\mathscr{T}_l(g_0)]A\|_1 < \epsilon/2$ for all $g\in\mathscr{U}_1$ and a neighborhood $\mathscr{U}_2$ of h_0 such that $\|[\mathscr{T}_r(h) - \mathscr{T}_r(h_0)]A\|_1 < \epsilon/2$ for all $h\in\mathscr{U}_2$. Then

$$\begin{aligned}&\|[\mathscr{T}_p(g,h) - \mathscr{T}_p(g_0,h_0)]A\|_1\\ &\quad= \|T(g)AT^+(h) - T(g_0)AT^+(h_0)\|_1\\ &\quad\leqslant\|T(g)AT^+(h) - T(g)AT^+(h_0)\|_1\\ &\quad\quad+\|T(g)AT^+(h_0) - T(g_0)AT^+(h_0)\|_1\\ &\quad=\|A[T^+(h)-T^+(h_0)]\|_1 + \|[T(g)-T(g_0)]A\|_1\\ &\quad=\|[\mathscr{T}_r(h)-\mathscr{T}_r(h_0)]A\|_1 + \|[\mathscr{T}_l(g)-\mathscr{T}_l(g_0)]A\|_1 < \epsilon.\end{aligned}$$

The continuity of the adjoint representation $\mathscr{T}_a$ is obtained from the restriction of the product representation $\mathscr{T}_p$ to the diagonal.

Similar representations may be defined on observable space $\mathscr{L}(H)$:

$$\tilde{\mathscr{T}}_l(g)\mathscr{O} = T(g)\mathscr{O}, \tag{5.6}$$

$$\tilde{\mathscr{T}}_r(g)\mathscr{O} = \mathscr{O}T^+(g), \tag{5.7}$$

$$\tilde{\mathscr{T}}_a(g)\mathscr{O} = T(g)\mathscr{O}T^+(g), \tag{5.8}$$

and

$$\tilde{\mathscr{T}}_p(g,h)\mathscr{O} = T(g)\mathscr{O}T^+(h). \tag{5.9}$$

However, these representations are not, in general, norm continuous. Thus, for example, let $H = L^2(R^3)$ be the square integrable functions on Euclidean 3-space, let G be the one-parameter group of rotations about the z axis, and let $[T(\theta)\psi](x,y,z) = \psi(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta, z)$ for $\psi\in L^2(H)$. The infinitesimal generator of T is the angular momentum operator L_z which is not bounded so that $\theta\mapsto T(\theta)$ is not norm continuous, i.e., $\|T(\theta)-I\|$ does not tend to zero as θ tends to zero.[11] This implies that the mapping $\theta\mapsto\tilde{\mathscr{T}}_l(g)I$ is not continuous at $\theta=0$. Similar examples can be given for the representations $\tilde{\mathscr{T}}_r$, $\tilde{\mathscr{T}}_a$, and $\tilde{\mathscr{T}}_p$. Because they are not norm continuous they are not weakly continuous.[14] They are, however, weak-* continuous; that is, the map

$g \mapsto \widetilde{\mathscr{T}}(g)\mathscr{O}$ is continuous when $\mathscr{L}(H)$ is given the weak-$*$ topology. Thus, fix $A \in \mathscr{I}_1(H)$ and $\epsilon > 0$. Because the maps $g \mapsto g^{-1}$ and $g \mapsto \mathscr{T}(g)A$ are continuous there exists a neighborhood $\mathscr{U}$ of g_0 such that $\|[\mathscr{T}(g^{-1}) - \mathscr{T}(g_0^{-1})]A\|_1 < \epsilon/\|\mathscr{O}\|$ for all $g \in \mathscr{U}$. Then

$$
\begin{aligned}
&|\langle\langle \widetilde{\mathscr{T}}(g)\mathscr{O}|A\rangle\rangle - \langle\langle \widetilde{\mathscr{T}}(h)\mathscr{O}|A\rangle\rangle| \\
&\quad = |\langle\langle \mathscr{O}|[\mathscr{T}(g^{-1}) - \mathscr{T}(g_0^{-1})]A\rangle\rangle| \\
&\quad \leqslant \|\mathscr{O}\|\cdot\|[\mathscr{T}(g^{-1}) - \mathscr{T}(g_0^{-1})]A\|_1 < \epsilon
\end{aligned}
$$

for all $g \in \mathscr{U}$.

We now inquire into the reducibility of these representations. The irreducible subrepresentations of the unitary representation $T: G \to \mathscr{L}(\mathrm{H})$ provide irreducible subrepresentations in $\mathscr{I}_1(H)$ and $\mathscr{L}(H)$. Let $H = \Sigma_{j\in J} \oplus H_j$ be a direct sum decomposition of H into finite-dimensional orthogonal, invariant, irreducible subspaces as guaranteed by the theorem of Gurevic.[4] Let P_j be the continuous self-adjoint projection operator from H to H_j. For each $(j', j'') \in J \times J$ define the operator $\mathscr{P}_{j'j''}$ on $\mathscr{I}_1(H)$ and the operator $\widetilde{\mathscr{P}}_{j'j''}$ on $\mathscr{L}(H)$ by

$$\mathscr{P}_{j'j''}A = P_{j'}AP_{j''}, \quad A \in \mathscr{I}_1(H), \tag{5.10}$$

$$\widetilde{\mathscr{P}}_{j'j''}\mathscr{O} = P_{j'}\mathscr{O}P_{j''}, \quad \mathscr{O} \in L(H), \tag{5.11}$$

They clearly satisfy the relations

$$\mathscr{P}_{j_1'j_1''}\mathscr{P}_{j_2'j_2''} = P_{j_1'j_1''}\delta_{j_1'j_2'}\delta_{j_1''j_2''}$$

and

$$\widetilde{\mathscr{P}}_{j_1'j_1''}\widetilde{\mathscr{P}}_{j_2'j_2''} = \widetilde{\mathscr{P}}_{j_1'j_1''}\delta_{j_1'j_2'}\delta_{j_1''j_2''}$$

and thus are projection operators. Furthermore, they do not increase the norm

$$\|\mathscr{P}_{j'j''}A\|_1 \leqslant \|A\|_1 \quad \forall A \in \mathscr{I}_1(H), \tag{5.12}$$

$$\|\widetilde{\mathscr{P}}_{j'j''}\mathscr{O}\| \leqslant \|\mathscr{O}\| \quad \forall \mathscr{O} \in \mathscr{L}(H), \tag{5.13}$$

and so they are norm continuous. The conditions TDS1, TDS2, and TDS3 are thus satisfied for the families $\mathscr{P}_{j'j''}$ and $\widetilde{\mathscr{P}}_{j'j''}$. If TDS4 is also satisfied, then they define topological direct sum decompositions of $\mathscr{I}_1(H)$ and $\mathscr{L}(H)$, respectively.

We show first that TDS4 is satisfied for the operators $\mathscr{P}_{j'j''}$ on $\mathscr{I}_1(H)$ when $\mathscr{I}_1(H)$ is given the norm topology. It then follows trivially that TDS4 is also satisfied if $\mathscr{I}_1(H)$ is given the weak topology. Let A be in $\mathscr{I}_1(H)$, and let $A = \Sigma_{i\in I} |\phi_i\rangle\lambda_i\langle\psi_i|$ be its polar decomposition. For any pair of finite subsets J' and J'' of J, let

$$A^{(J',J'')} = \sum_{(j',j'')\in J'\times J''} \mathscr{P}_{j'j''}A = \sum_{(j',j'')\in J'\times J''} \mathscr{P}_{j'}A\mathscr{P}_{j''}$$

and let $\psi^{J'} = \Sigma_{j'\in J'}\mathscr{P}_{j'}\psi$ for ψ in H. Choose N so that $\Sigma_{i=N+1}^{\infty} 2\lambda_i < \epsilon/2$ and choose a finite subset K of J so that $N\lambda_{\max}(\|\psi_i - \psi_i^{J''}\|_H + \|\phi_i - \phi_i^{J'}\|_H) < \epsilon/2$ for all finite subsets J' and J'' of J containing K and for all $i \leqslant N$, where $\lambda_{\max} = \sup_{i\in I}\lambda_i$. Then for all $B \in \mathscr{L}(H)$ with $\|B\| = 1$ we have

$$
\begin{aligned}
&|\langle\langle B|A\rangle\rangle - \langle\langle B|A^{J'J''}\rangle\rangle| \\
&\quad = \left|\sum_{i\in I}\lambda_i(\langle\psi_i|B|\phi_i\rangle - \langle\psi_i^{J''}|B|\phi_i^{J'}\rangle)\right| \\
&\quad \leqslant \sum_{i\in I}\lambda_i|\langle\psi_i|B|\phi_i\rangle - \langle\psi_i^{J''}|B|\phi_i^{J'}\rangle| \\
&\quad = \sum_{i\in I}\lambda_i|\langle\psi_i - \psi_i^{J''}|B|\phi_i\rangle + \langle\psi_i^{J''}|B|\phi_i - \phi_i^{J'}\rangle| \\
&\quad \leqslant \sum_{i\in I}\lambda_i(\|\psi_i - \psi_i^{J''}\|_H + \|\phi_i - \phi_i^{J'}\|_H) \\
&\quad \leqslant \sum_{i=1}^{N}\lambda_i(\|\psi_i - \psi_i^{J''}\|_H + \|\phi_i - \phi_i^{J'}\|_H) + \sum_{i=N+1}^{\infty} 2\lambda_i \\
&\quad \leqslant \epsilon
\end{aligned}
$$

for all finite subsets J' and J'' of J containing K. This shows, by virtue of (3.5), that

$$\sum_{(j',j'')\in J\times J} \mathscr{P}_{j'j''}A$$

converges to A in $\mathscr{I}_1(H)$ norm for all A in $\mathscr{I}_1(H)$.

On $\mathscr{L}(H)$ the situation is somewhat different. Since the norm closure of the set of finite rank operators is the set of compact operators and since for every $\mathscr{O} \in \mathscr{L}(H)$ and every finite subset $J' \times J'' \subset J \times J$ the operator

$$\widetilde{\mathscr{P}}_{J'J''}\mathscr{O} = \sum_{(j',j'')\in J'\times J''} \widetilde{\mathscr{P}}_{j'j''}\mathscr{O} = P_{J'}\mathscr{O}P_{J''}$$

has finite rank, TDS4 will not be satisfied if H is infinite-dimensional and $\mathscr{L}(H)$ is given the norm topology. However, TDS4 is satisfied if $\mathscr{L}(H)$ is given the weak-$*$ topology. Thus choose $A \in \mathscr{I}_1(H)$. Then

$$
\begin{aligned}
&|\langle\langle \widetilde{\mathscr{P}}_{J'J''}\mathscr{O}|A\rangle\rangle - \langle\langle \mathscr{O}|A\rangle\rangle| \\
&\quad = |\langle\langle \mathscr{O}|(\mathscr{P}_{J'J''} - I)A\rangle\rangle| \\
&\quad \leqslant \|\mathscr{O}\|\cdot\|(\mathscr{P}_{J'J''} - I)A\|_1,
\end{aligned}
$$

which converges to zero since $\|(\mathscr{P}_{J'J''} - I)A\|_1$ converges to zero.

Let $\mathscr{I}_1^{j'j''}$ be the range of $\mathscr{P}_{j'j''}$ and let $\mathscr{L}^{j'j''}$ be the range of $\widetilde{\mathscr{P}}_{j'j''}$. To review, $\mathscr{I}_1(H)$ and $\mathscr{L}(H)$ have the direct sum decompositions

$$\mathscr{I}_1 = \sum_{(j',j'')\in J\times J} \oplus \mathscr{I}_1^{j'j''}, \tag{5.14}$$

$$\mathscr{L}(H) = \sum_{(j',j'')\in J\times J} \oplus \mathscr{L}^{j'j''}. \tag{5.15}$$

Each $\mathscr{I}_1^{j'j''}$ is a finite-dimensional subspace of $\mathscr{I}_1$ which is invariant under the representations $\mathscr{T}_1$, $\mathscr{T}_r$, $\mathscr{T}_a$, and $\mathscr{T}_p$. Similarly each $\mathscr{L}^{j'j''}$ is a finite-dimensional subspace of $\mathscr{L}(H)$, which is invariant under the representations $\widetilde{\mathscr{T}}_1$, $\widetilde{\mathscr{T}}_r$, $\widetilde{\mathscr{T}}_a$, and $\widetilde{\mathscr{T}}_p$. The subspaces $\mathscr{I}_1^{j'j''}$ and $\mathscr{L}^{j'j''}$ are obviously irreducible under the product representations $\mathscr{T}_p$ and $\widetilde{\mathscr{T}}_p$, respectively. Since they are finite-dimensional, they may be decomposed into irreducible subspaces for each of the other representations using familiar finite-dimensional techniques. (The irreducible subspaces thus obtained consist of the standard irreducible tensorial operators.) Consequently, state space $\mathscr{I}_1(H)$ and observable space $\mathscr{L}(H)$ are completely reducible under the left, right, adjoint, and product representations of the compact and Hausdorff group G.

The set of operators in $\mathscr{L}^{j'j''}$ is identical to the set of operators in $\mathscr{I}^{j'j''}$, and $\mathscr{L}^{j'j''}$ and $\mathscr{I}^{j'j''}$ are isomorphic vector spaces. The fact that the nonisomorphic vector spaces $\mathscr{L}(H)$ and $\mathscr{I}_1(H)$ are the direct sums of these isomorphic vector spaces is possible because of the nonuniqueness of infinite direct sums as explained in Sec. IV.

VI. REPRESENTATION THEORY ON THE SPACES OF THE ALGEBRAIC APPROACH

The algebraic approach[15,16] is a quantum mechanical formalism which is closely related in spirit to the Liouville representation approach. It differs from the Liouville representation in that the space of states is defined to be the set of *all* positive linear functionals ϕ on the set of observables such that $\phi(I) = 1$. Although the spaces of the algebraic approach are defined abstractly, a concrete realization of the axiomatic system is provided by representing each observabe by a self adjoint element of $\mathscr{L}(H)$ for some Hilbert space H. The states are then represented by elements in the dual space $\mathscr{L}^*(H)$. This dual space is a topological direct sum[10]

$$\mathscr{L}^*(H) = \mathscr{I}_1(H) \oplus \mathrm{Com}(H)^\perp$$

of the trace class operators $\mathscr{I}_1(H)$ and the space $\mathrm{Com}(H)^\perp$ of linear functionals which vanish on the compact operators. Thus, whenever H is infinite-dimensional, the set of states used in the algebraic approach is much larger than the set of states used in the Liouville representation. In this section we consider the difference this enlarged state space has on the group representations.

The representations $\tilde{\mathscr{T}}_l$, $\tilde{\mathscr{T}}_r$, $\tilde{\mathscr{T}}_a$, and $\tilde{\mathscr{T}}_p$ defined in Eqs. (4.6)–(4.9) still exist on $\mathscr{L}(H)$. These may be used to define representations on $\mathscr{L}^*(H)$ by

$$\tilde{\tilde{\mathscr{T}}}_l(g) = \tilde{\mathscr{T}}_l^\dagger(g^{-1}),$$
$$\tilde{\tilde{\mathscr{T}}}_r(g) = \tilde{\mathscr{T}}_r^\dagger(g^{-1}),$$
$$\tilde{\tilde{\mathscr{T}}}_a(g) = \tilde{\mathscr{T}}_a^\dagger(g^{-1}),$$
$$\tilde{\tilde{\mathscr{T}}}_p(g, h) = \tilde{\mathscr{T}}_p^\dagger(g^{-1}, h^{-1}),$$

where $\dagger$ refers to the adjoint operation.

In the algebraic approach there are two natural topologies to consider on $\mathscr{L}(H)$ and $\mathscr{L}^*(H)$. On $\mathscr{L}(H)$ it is natural to consider the norm topology and the weak (Banach space) topology. Both of these topologies are finer than the weak-$*$ topology which is natural in the Liouville representation. On $\mathscr{L}^*(H)$ it is natural to consider the norm topology and the weak-$*$ topology. We must now ask if the representations $\tilde{\mathscr{T}}$ and $\tilde{\tilde{\mathscr{T}}}$ are continuous in any of these topologies. We have already seen that the representations $\tilde{\mathscr{T}}$ on $\mathscr{L}(H)$ are not norm-continuous for many physically significant groups such as one-parameter rotation groups. These one-parameter subgroups cannot be weakly continuous either since weak continuity implies norm continuity.[14] This in turn implies that $\tilde{\tilde{\mathscr{T}}}$ on $\mathscr{L}^*(H)$ cannot be weak-$*$ continuous which implies that $\tilde{\tilde{\mathscr{T}}}$ cannot be norm continuous. In our opinion the failure of these representations to be continuous in physically significant topologies represents, for certain applications, a significant disadvantage of the algebraic approach relative to the Liouville representation approach in which the corresponding representations are continuous.

Since the representations $\tilde{\mathscr{T}}$ and $\tilde{\tilde{\mathscr{T}}}$ are not, in general, continuous, it is difficult to proceed with a general representation theory. It is clear, moreover, that even when $\tilde{\mathscr{T}}$ is continuous the general observable $\mathscr{O} \in \mathscr{L}(H)$ cannot be expanded in terms of irreducible tensorial operators in either a norm convergent or a weak convergent sense. This is because each irreducible tensorial operator is of finite rank. The limit of such operators in either the norm or the weak topology is a compact operator. Thus, unless observable $\mathscr{O}$ is compact, it cannot be expanded in terms of irreducible tensorial operators.

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