Optimal paths for thermodynamic systems: The ideal Otto cycle

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We apply the method of optimal control theory to determine the optimal piston trajectory for successively less idealized models of the Otto cycle. The optimal path has significantly smaller losses from friction and heat leaks than the path with conventional piston motion and the same loss parameters. The resulting increases in efficiency are of the order of 10%.

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I. INTRODUCTION

Equilibrium thermodynamics concerns itself with limiting cases of possible processes. Thus it can be used to place an upper bound on the performance of heat engines, but not to analyze processes taking place at finite rates. In order to analyze these we need more than a knowledge of the equilibrium states; we also need to consider the details of the irreversibilities and the time or rate parameters (evolution) of the system.

In recent work[1-6] in finite-time thermodynamics the irreversibilities considered have been those due to a finite rate of heat transfer between heat reservoirs and the working fluid. The objective has been to determine the trajectories which optimize some objective function, such as efficiency or power.

In this work we find the optimum time path for a thermodynamic system with friction and a finite rate of heat leakage. For our system we choose an internal-combustion Otto cycle. In practice, thermal and friction losses reduce the efficiency of these engines by about 40%.[7,8] We have also considered the effects of finite bounds on the acceleration and deceleration.

The internal combustion engine has been the object of an enormous amount of engineering research. However, it appears that very little attention has been paid to how the piston motion can affect the performance of the engine. We have therefore concentrated our attention on the piston motion and have evaluated the improvements that could be realized by optimizing this motion.

Although our model is far simpler than an actual internal combustion engine, it serves several purposes. First, we use it to illustrate the optimization of the time path of a system. Second, the model leads to a qualitative understanding of how engine losses can be reduced. This insight will, we hope, guide more detailed and realistic models. Finally, we do evaluate the amount of improvement that might be achieved by optimizing the piston motion for an engine with losses representative of those encountered in actual internal combustion engines.

Previous work in finite-time thermodynamics has generally optimized either power or efficiency. In practice, however, some compromise between these objectives is made. Since we wish to have fairly realistic operating conditions, we have fixed the total cycle time and fuel consumed per cycle. Under these constraints maximizing efficiency, effectiveness, and average power are all identical.

In Sec. II we describe the losses usually encountered in internal combustion engines and present our model of these. We then describe the conventionally operating engine with which our optimized operation will be compared.

The procedure used to find the optimum trajectory is described in Sec. III. The computational results for both the optimal and conventional trajectories are given in Sec. IV.

In Sec. V we sum up and evaluate the improvements in internal combustion engine performance that could result from piston trajectory optimization.

II. DESCRIPTION OF MODEL

A. Loss terms

The following is a description of the major sources of irreversibilities in internal combustion engine operation. It is based on information taken from Taylor's monograph.[7] The models that we employ for the losses are considerably simplified. However, they do reproduce the qualitative behavior and the total magnitude of these losses.

1. Friction

Friction typically dissipates about 20% of the power developed by the engine. Of this about 75% is due to the friction of the piston rings on the cylinder walls and 25% is in the crankshaft bearings. The latter contribution is assumed to be independent of the piston motion. For the former contribution we assume a friction force linear in velocity, i.e.,

Friction force $= \alpha V$,

corresponding to a well-lubricated system. Thus the work consumed by friction in a stroke taking time t is

$$W_f = \int_0^t \alpha v^2 \, dt'. \tag{1}$$

Due to greater pressure on the piston the value of α is usually about twice as large on the power stroke as on the other strokes. We assume that the heat dissipated by the friction is not returned directly to the working fluid.

2. Pressure drop

There is an additional friction-like loss term on the intake stroke. This is due to the pressure differential that develops, due to viscosity, as the gas flows through the inlet valve. The pressure differential is proportional to velocity, so it may be included in the friction term for the intake stroke.

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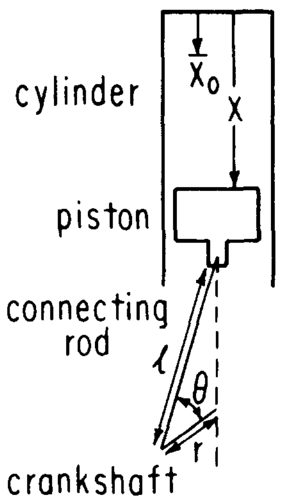

FIG. 1. Conventional piston linkage.

The force developed is about twice that for the rubbing friction.[7] On the expansion stroke the pressure drop is negligible.

3. Heat leak

Losses due to heat transfer from the working fluid to the cylinder walls typically cost about 12% of the total power. The percentage of the heat of combustion transferred to the cooling system is about 30%. The difference occurs because much of the heat lost through the cylinder walls would otherwise have been expelled with the exhaust. The heat transfer expression used here assumes that the rate is linear in the inside surface area of the cylinder and in the difference between the temperature T of the working fluid and that of the wall T_w. T_w is assumed to be constant. For heat conduction coefficient κ and cylinder diameter b, the rate of heat leak at position X (see Fig. 1) is

$$\dot{Q} = \kappa\pi b\left(\tfrac{1}{2}b + X\right)(T - T_w). \quad (2)$$

The effect of heat transfer is only important on the power stroke. The rate is negligible on the other strokes since $T - T_w$ is much smaller on them.

4. Time loss

Losses due to beginning the expansion stroke while combustion is still taking place amount to about 6% of the reversible work. Since the burning velocity is strongly dependent on the piston motion, the time loss has little dependence on the motion.[7]

5. Exhaust blowdown

In a conventional engine losses due to opening the exhaust valve before completion of the expansion stroke cost less than 2% of the total power. These losses are not included here.

B. Conventional engine

The engine parameters used in these calculations are listed in Table I. The losses due to bearing friction and time loss are simply subtracted from the total work per cycle since these are independent of the piston trajectory. The friction and heat leak coefficients are calculated so as to produce losses of the correct magnitude[7,8] for the cycle when Eqs. (1) and (2) are used with the conventional piston motion. These calculations are described below.

Figure 1 shows a typical piston to crankshaft linkage. It is a simple matter to show that the equation of motion for the piston is[9]

$$\dot{X} = \frac{2\pi\Delta X}{\tau}\sin\theta\left\{1 + \frac{r}{l}\cos\theta\left[1 - (r/l)^2\sin^2\theta\right]\right\}^{-1/2}, \quad (3)$$

where $\Delta X = 2r$ and $\theta = 4\pi t/\tau$. τ is the cycle time (two crankshaft revolutions) and $X = X_0$ when $t = 0$. If $r/l = 0$, the motion would be purely sinusoidal. Typically r/l is between 0.16 and 0.40[9]; for these calculations we used 0.25. Varying the value of r/l had little effect on the results.

TABLE I. Engine parameters.[a]

	Compression stroke	Power stroke
Mechanical parameters		
compression ratio = 8		
$X_0 = 1$ cm, $\Delta X = 7$ cm		
cylinder bore, $b = 7.98$ cm		
cylinder volume, $V = 400$ cm^3		
cycle time $\tau = 33.3$ msec corresponding to 3600 rpm		
Thermodynamic parameters		
Initial temperature	333 K	2795 K
No. of moles of gas	0.0144	0.0157
const. vol. heat capacity	2.5 R	3.35 R
cylinder wall temp., $T_w = 600$ K		
reversible work per cycle, $W_R = 435.7$ J		
reversible power, $W_R/\tau = 13.1$ kW		
Loss terms		
friction coefficient, $\alpha = 12.9$ kg sec^{-1}		
heat leak coefficient, $\kappa = 1305$ kg deg^{-1} sec^{-3}		
work lost per cycle to time loss and bearing friction, $W_B = 50$ J		

[a]The parameters are based on data from Ref. 7.

The friction coefficient α is determined as follows: We substitute Eq. (3) into (1), set r/l equal to zero (purely sinusoidal motion) and integrate from 0 to $\frac{1}{4}\tau$ (one stroke). This gives the losses for one stroke

$$W_f = [\alpha\pi^2(\Delta X)^2]/2\tau.$$

Now if α is the friction coefficient of the exhaust and compression strokes, then the coefficient on the power stroke is 2α, and on the intake stroke (including pressure drop) it is 3α. Thus the losses for the full cycle are

$$W_f = (7\alpha\pi^2\Delta X^2)/2\tau.$$

We then set $W_f = 0.15 W_R$ and use the parameters from Table I to find $\alpha = 12.9$ kg sec^{-1}.

The Eqs. (1), (2), and (3) determine the time evolution of the system. They can be solved numerically to calculate the total work per cycle. The heat leak coefficient κ was determined by trying various values in the equations until the total work lost was about 10% of the reversible work. It can be roughly obtained as follows: Letting $\bar{Q}$ and $\bar{\eta}$ be the average rate of heat leak and efficiency, we have for the work lost due to the heat leak.

$$W_Q \approx \bar{\eta}\bar{Q}\tfrac{1}{4}\tau.$$

Then from Eq. (2)

$$\bar{Q} \approx \kappa(\tfrac{1}{2}b + \bar{X})(\pi b)(\bar{T} - T_w),$$

with $\bar{X} = 4.5$ cm and $\bar{T} = 1800$ K. Combining these and using the values from Table I we find that $W_Q/W_R = 0.1$ and $\bar{\eta} = 0.157$ corresponds to $\kappa = 1305$ kg deg^{-1} sec^{-3}.

III. OPTIMIZATION PROCEDURE

The optimization problem we set here is finding the maximum work per cycle for fixed fuel consumption and total cycle time. Thus the only difference between the optimized engine and the conventional one is in the piston motion. The procedure consists of finding the optimal trajectory on each stroke as a function of the time spent on that stroke and then optimizing the distribution of time among the strokes.

Since heat transfer effects are negligible on the nonpower strokes, optimization of these strokes is relatively simple. The three strokes can be treated together with a fixed total time; this is done in Sec. III A.

The power stroke is more difficult. Some portions of the trajectory lie on boundaries determined by the limits placed on the acceleration. We shall see that the rest of the solution is given in terms of a fourth-order differential system with boundary conditions given at both end points. To complicate matters the points at which the solution leaves the boundaries are not known *a priori*.

Instead of attacking the fully elaborated problem directly, we approached it in several steps. First, we considered a problem with no constraints on either the acceleration or on the time to complete the power stroke. It turns out that the optimal time for the power stroke is finite, anyway. The solution to this problem requires solving a second-order differential system with one initial condition and one final condition. We were able to find an efficient method of guessing the unknown initial condition. Then, using the IBM Continuous Systems Modeling Program (CSMP), we solved the equations numerically. The final condition that resulted was compared to the known final condition and used to refine the guess. The details of the entire procedure are described in Sec. III B.

Next, we restored the time constraint and found the Euler-Lagrange equations that describe the trajectory. These formed a fourth-order system with two initial and two final conditions. Using the results from the time unconstrained case as a starting point we solved these equations and combined them with results for the nonpower strokes to get the optimal trajectory for the entire cycle. We describe this procedure in Sec. III C.

Finally, we placed limits on the acceleration. The trajectory is given by a fourth-order system connecting two boundary solutions. We now had three initial conditions and three final conditions, knowing the two extra boundary conditions is balanced out by lack of knowledge of the two points at which the solution passes from the boundaries to the interior. In Sec. III D we give the details of this calculation.

In the sections that follow we deal with finding extremal values of integral expressions subject to inequality constraints. The classical calculus of variations in the form of optimal control theory can be extended to deal with such problems.[10] We have used this theory wherever necessary. A statement of the optimal control problem and necessary conditions for the optimal path are given in the Appendix.

A. Nonpower strokes

First, we need to minimize the friction losses on a single stroke completed in time t_1. From Eq. (1)

$$W_f = \int_0^{t_1} \alpha \dot{X}^2 \, dt.$$

If we place no limits on the acceleration, we can use the calculus of variations to find the trajectory which minimizes W_f. This is given by

$$\dot{X} = \text{constant} = \Delta X / t_1.$$

For the case in which the acceleration is constrained to lie between $-a_m$ and a_m, it is obvious that the trajectory will be as follows: Start at $v = 0$, accelerate at the maximum rate until some time t_a, run at constant velocity $v = a_m t_a$ until $t = t_1 - 2t_a$, then decelerate at the maximum rate for the rest of the stroke until $v = 0$. Optimal control theory can be used to show that this is indeed the solution.

We find t_a and the friction losses per stroke as follows. Requiring that the piston move a distance ΔX in time t_1, we have

$$\Delta X = a_m t_a^2 + a_m t_a (t_1 - 2t_a).$$

Solving for t_a we get

$$t_a = \tfrac{1}{2} t_1 (1 - y_1), \tag{4}$$

where

$$y_1 = \left(1 - \frac{4\Delta X}{a_m t_1^2}\right)^{1/2}.$$

We can now get the friction losses per stroke by integrating Eq. (1):

$$W_f = \alpha\left(2\int_0^{t_a} (a_m t)^2\, dt + \int_{t_a}^{t_1 - t_a} (a_m t_a)^2\, dt\right),$$

to get

$$W_f = \frac{\alpha a_m^2}{12} t_1^3 (1 + 2y_1)(1 - y_1)^2, \tag{5}$$

where Eq. (4) has been used.

We now wish to find the time t_1 which minimizes the total friction losses for the three nonpower strokes. The friction coefficient α is the same on the exhaust and compression strokes. Therefore the time t_1 spent on each of these strokes is the same. The friction coefficient on the intake stroke is 3α, let the time spent on this stroke be t_2. We can now use Eq. (5) to write down the total losses on the nonpower strokes:

$$W_f = \tfrac{1}{6}\alpha a_m^2 \left[t_1^3 (1 - y_1)^2 (1 + 2y_1) + \tfrac{3}{2} t_2^3 (1 - y_2)^2 (1 + 2y_2)\right]. \tag{6}$$

We now let the total time for the three strokes be $t = 2t_1 + t_2$ and set $\partial W_f / \partial t_2 = 0$. After rearranging we get

$$t_1^2 (1 - y_1)^2 = 3 t_2^2 (1 - y_2)^2. \tag{7}$$

For a given value of t this equation can be solved numerically to get t_1 and t_2.

For the case where $a_m \to \infty$, Eq. (7) becomes

$$t_2 = (\sqrt{3}) t_1,$$

and the total friction losses, from Eq. (6), are

$$W_f = \alpha (2 + \sqrt{3})^2 (\Delta X)^2 / t.$$

B. Power stroke—time unconstrained case

If the problem is parameterized in terms of the piston position X rather than the time t, it is no longer necessary to specify a time duration for the power stroke. The optimization problem becomes

$$\text{maximize } W_p = \int_{X_0}^{X_f} \left(\frac{NRT}{X} - \alpha v\right) dX,$$

subject to the constraint on the internal energy

$$\frac{dT}{dx} \equiv T' = \frac{-1}{NC}\left[\frac{NRT}{X} + \frac{\pi b \kappa}{v}\left(\frac{b}{2} + X\right)(T - T_w)\right]. \tag{8}$$

We assume that the heat capacity C is independent of temperature. Since $v(X)$ might be discontinuous at the endpoints, this problem shold be treated by optimal control theory.[10] The Hamiltonian is

$$H = \frac{NRT}{X} - \alpha v - \frac{\lambda}{NC}\left[\frac{NRT}{X} + \left(\frac{\pi b \kappa}{v}\right)\left(\frac{b}{2} + X\right)(T - T_w)\right].$$

The canonical equations are Eq. (8) and

$$\lambda' = -\frac{NR}{X} + \frac{\lambda R}{CX} + \frac{\lambda \kappa \pi b}{NCv}\left(\frac{b}{2} + X\right).$$

The maximum principle becomes

$$v = \left[\frac{\kappa \pi b \lambda (T - T_w)}{\alpha C N}\left(\frac{b}{2} + X\right)\right]^{1/2}. \tag{9}$$

We now have a second-order system with boundary conditions $T(X_0) = T_0$, $\lambda(X_f) = 0$.

λ_0 can be estimated in the following manner.[11] Let J_j be the maximum work that can be done in expanding from X_j to X_f

$$J_j = \underset{[x_j, x_f]}{v\max} \int_{X_j}^{X_f} F\, dx,$$

where

$$F = NRT/X - \alpha v,$$

and $\underset{[X_j, X_f]}{v\max}$ indicates that the maximization is to be with respect to the function v on the interval $[X_j, X_f]$. We can then write

$$J_0 = v \underset{[X_0, X_1]\, \Omega_1}{\max},$$

where we define Ω_1 by

$$\Omega_1 = \int_{X_0}^{X_1} F\, dx + J_1.$$

J_1 can also be expressed as the product of the reversible work done in an adiabatic expansion and the effectiveness ϵ_1 of the process starting from X_1:

$$J_1 = \epsilon_1 NCT\left[1 - (X_1/X_f)^{R/C}\right].$$

For small values of $\Delta X \equiv X_1 - X_0$ we can write

$$\Omega_1 = \left(\frac{NRT_0}{X_0} - \alpha v_0\right)\Delta X + \epsilon_1 NRCT_1\left[1 - \left(\frac{X_0 + \Delta X}{X_f}\right)^{R/C}\right].$$

From Eq. (8) we can write

$$T_1 = T_0 - \frac{\Delta X}{NC}\left[\frac{NRT_0}{X_0} + \frac{\kappa \pi b}{v_0}\left(\frac{b}{2} + X_0\right)(T_0 - T_w)\right].$$

Substituting this in the expression for Ω_1 and setting $\partial \Omega_1 / \partial v_0 = 0$ we get

$$v_0 = \left\{\epsilon \frac{\kappa}{\alpha}\left(\frac{b}{2} + X_0\right)(\pi b)(T_0 - T_w)\left[1 - \left(\frac{X_0}{X_f}\right)^{R/C}\right]\right\}^{1/2},$$

or, using (9)

$$\lambda_0 = \epsilon NC\left[1 - (X_0/X_f)^{R/C}\right].$$

We know ϵ will be somewhat larger than the effectiveness of 0.85 realized with the sinusoidal piston motion (this is for the power stroke, not the whole cycle), so this equation provides a good means to get a reasonable first guess for λ_0. We then improve the value of λ_0 iteratively until $\lambda(X_f)$ is reasonably close to zero.

C. Power stroke with time constraint

We wish to maximize the amount of work done on the power stroke in time t'. This is given by the integral

$$W_P = \int_0^{t'}\left(\frac{NRT\dot{X}}{X} - \alpha \dot{X}^2\right) dt, \tag{10}$$

with the constraint

$$\dot{T} = \frac{-1}{NC}\left[\frac{NRT\dot{X}}{X} + \kappa \pi b\left(\frac{b}{2} + X\right)(T - T_w)\right]. \tag{11}$$

The Lagrangian for the problem is

$$L = \frac{NRT\dot{X}}{X} - \alpha \dot{X}^2 + \lambda \left[\dot{T} + \frac{RT\dot{X}}{CX} + \frac{\kappa \pi b (b/2 + X)(T - T_w)}{NC} \right].$$

From this we obtain the Euler-Lagrange equations which can be rearranged to yield

$$\dot{X} = v, \tag{12}$$

$$\dot{v} = \frac{\kappa \pi b}{2\alpha NC} \left\{ (T_w - T)\left(\frac{NR}{X}\right)\left(\frac{b}{2} + X\right) + \lambda \left[\frac{RT_w}{CX}\left(\frac{b}{2} + X\right) - (T - T_w) \right] \right\}, \tag{13}$$

$$\dot{\lambda} = \frac{NRv}{X}\left(1 + \frac{\lambda}{NC}\right) + \frac{\lambda \kappa \pi b}{NC}\left(\frac{b}{2} + X\right). \tag{14}$$

Equations (11)–(14) form a fourth-order system of differential equations. If they could be solved they would yield the maximum value of W_P as a function of t'. It would then be possible to combine this result with (7) to find the values of t and t', subject to $\tau = t + t'$, that maximizes the work done in the entire cycle.

In solving these equations, there are four boundary conditions to be satisfied. These are

$$X(0) = X_0, \quad X(t') = X_f, \quad T(0) = T_0,$$

and

$$\left.\frac{\partial L}{\partial \dot{T}}\right|_{t=t'} = \lambda(t') = 0.$$

This last condition occurs because we do not wish to fix the final temperature.[10]

The numerical solution of the Euler-Lagrange equations requires guessing two initial conditions (λ_0 and v_0) and attempting to "hit" the final conditions $[X(t'), \lambda(t')]$. This would have to be done for enough values of t' to be able to do the time-distribution part of the problem. The amount of computation required was kept fairly reasonable by means of the following procedure.

(i) The time unconstrained problem was solved in order to get a starting point. The initial velocity for this case was taken as the first guess for v_0 and the initial acceleration was used with Eq. (13) to guess λ_0. (ii) Equations (11)–(14) were then solved numerically until the condition $X = X_f$ was met, rather than over a specified time interval. λ_0 was then varied until λ was reasonably close to zero at this point. The result was the optimal trajectory on the power stroke for some time t'. (iii) The trajectory resulting from (ii) would be for the longest t' of interest since it would be close to the solution for the time unconstrained case. Solutions for shorter t' were obtained by increasing v_0, using the value of λ_0 from the previous solution and then repeating step (ii). (iv) As each trajectory was obtained the total work was obtained by using Eq. (6). Step (iii) was repeated until the maximum was passed.

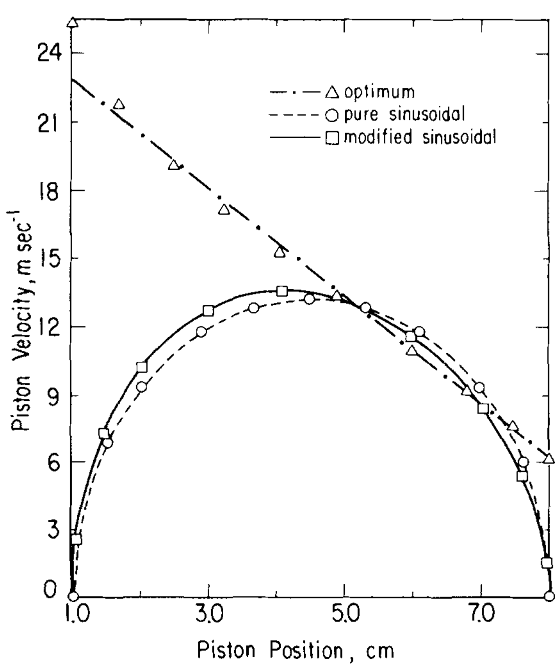

FIG. 2. Piston motion on the power stroke.

D. Limited acceleration

We will now require the velocity to be zero at both end points and constrain the acceleration to lie within finite limits. Because we wish to apply inequality constraints to the acceleration, we will treat this as an optimal control problem.[10]

The control problem is the same as that of the previous section, but now the integrand in Eq. (10) is regarded as depending implicitly on the acceleration a. The dependence of the state variables T, X, and v on the control variable a is given by Eqs. (11), (12), and

$$\dot{v} = a. \tag{15}$$

We also require that $-a_m \leqslant a \leqslant a_m$.

The Hamiltonian for this problem is

$$H = \frac{NRTv}{X} - \alpha v^2 - \frac{\lambda_1}{NC}\left[\frac{NRTv}{X} + \kappa \pi b \left(\frac{b}{2} + X\right)(T - T_w) \right] + \lambda_2 v + \lambda_3 a.$$

The canonical equations conjugate to (11), (12), and (15) are

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial T} = \frac{NRv}{X}\left(\frac{\lambda_1}{NC} - 1\right) + \frac{\lambda_1}{NC}\kappa \pi b \left(\frac{b}{2} + X\right), \tag{16}$$

TABLE II. Parameters for different cases.

Case	α kg sec^{-1}	κ k_g deg^{-1} sec^{-3}	τ msec	rpm
I	12.9	1305	33.33	3600
II	7.50	2350	33.33	3600
III	17.2	650	33.33	3600
IV	12.9	1305	25.00	4800
V	12.9	1305	50.00	2400

TABLE III. Results for unconstrained acceleration.

Case	v_{max} m sec^{-1}	t' msec	W_P J	W_T J	W_f J	W_Q J	Q J	W_Q/Q	T_f K	ϵ
I										
conv.	13.6	8.33	503	276	67	43	224	0.19	1095	0.633
opt.	25.4	5.48	518	307	58	21	156	0.14	1200	0.705
II										
conv.	13.6	8.33	480	273	39	74	340	0.22	900	0.627
opt.	44.1	3.40	518	321	43	22	171	0.13	1170	0.737
III										
conv.	13.6	8.33	518	275	89	22	126	0.17	1270	0.631
opt.	16.9	6.52	525	302	71	13	100	0.13	1310	0.693
IV										
conv.	18.1	6.25	507	264	89	33	178	0.19	1180	0.606
opt.	26.2	4.74	517	294	73	19	138	0.14	1240	0.675
V										
conv.	9.1	12.5	489	278	44	63	301	0.21	960	0.638
opt.	25.0	6.10	518	319	45	22	170	0.13	1170	0.732

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial X} = \frac{NRTv}{X^2}\left(1 - \frac{\lambda_1}{NC}\right) + \frac{\lambda_1}{NC}\kappa\pi b(T - T_w), \quad (17)$$

$$\dot{\lambda}_3 = -\frac{\partial H}{\partial v} = 2av - \lambda_2 - \frac{NRT}{X}\left(1 - \frac{\lambda_1}{NC}\right). \quad (18)$$

From the maximum principle, the condition for an interior maximum is

$$0 = \frac{\partial H}{\partial a} = \lambda_3.$$

If this holds for more than isolated points we also have

$$\dot{\lambda}_3 = 0.$$

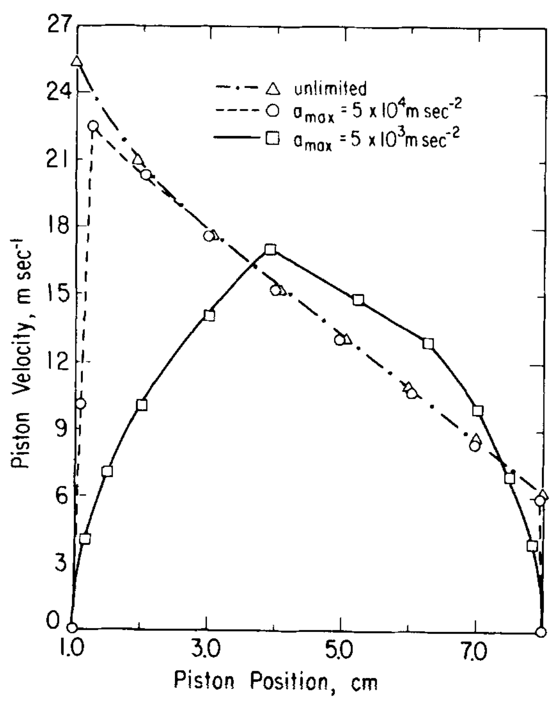

FIG. 3. Trajectories for various values of a_{max}.

Then if we eliminate λ_2 between Eqs. (17) and (18), we get the same set of equations as was obtained for the case of unlimited acceleration.

On this basis we can conclude that the trajectory is the one that we might have intuitively expected, namely, two boundary segments (maximum acceleration and maximum deceleration) connected by a segment which satisfies the system of equations of Sec. III C.

The solutions were obtained in a manner similar to that described in the previous section but with several important differences. These were: (a) Instead of varying λ_0 and "shooting" for $\lambda(t')$, we guessed the final temperature and tried to "hit" T_0 by solving the equations backwards. This turned out to give much faster convergence. (b) Instead of changing v_0 to get solutions for various values of t', we now changed the amount of time spent on the maximum deceleration segment. In effect this meant that we were varying the velocity at the end of the interior segment. (c) On the constant acceleration segment the velocity is related to the piston position by

$$v = [2a_m(X - X_0)]^{1/2}.$$

When this condition was met the solution was switched from the interior segment to the maximum acceleration segment. This replaces the condition $X = X_f$ in step (b).

IV. RESULTS

A. Piston trajectories

Figure 2 shows the piston velocity plotted versus the piston position on the power stroke. The plots shown are for the optimal trajectory with no constraints on the acceleration, the purely sinusoidal motion, and the modified sinusoidal motion described in Sec. II B.

The modified sinusoidal motion reaches a slightly higher peak velocity (13.6 m sec^{-1} versus 13.2 m sec^{-1}) than the pure sinusoidal motion. The peak for the modified motion is at 4.1 cm rather than 4.5 cm. As a result, the modified motion has higher velocities in the region where the temperature is highest. This results in an increase of ~1.5% in the

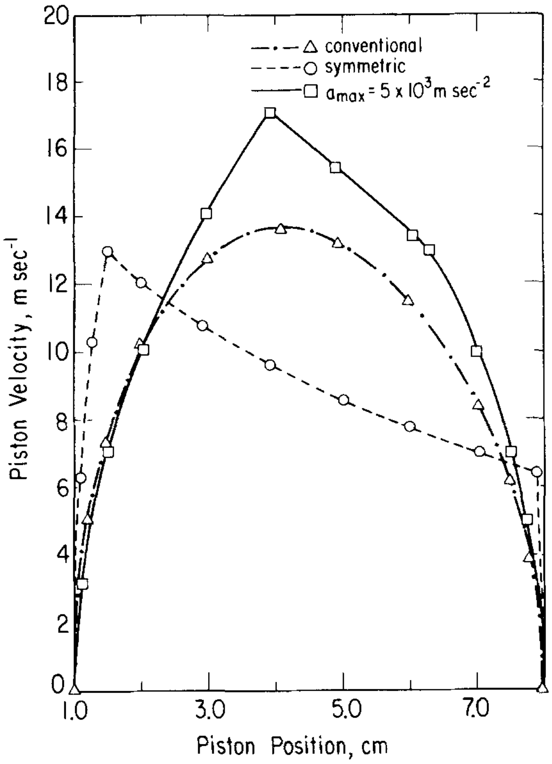

FIG. 4. Comparison of conventional, symmetric, and optimized trajectories.

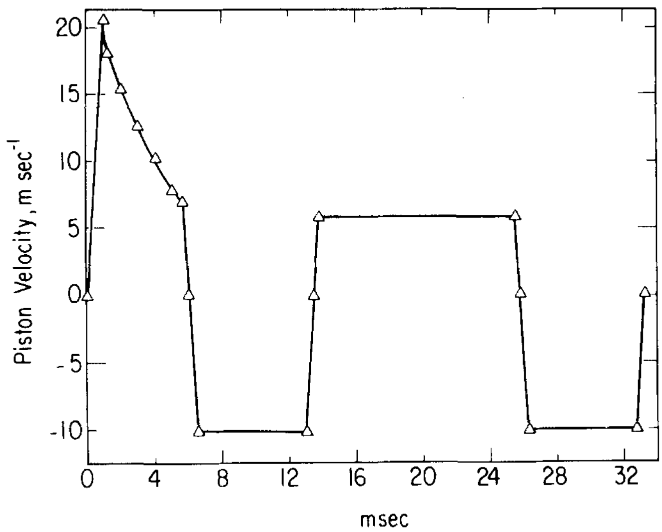

FIG. 5. Trajectory for the full cycle with $a_{max} = 2\times10^4$ m sec^{-2}.

friction losses, a reduction of ~6% in the heat leak, and an increase in effectiveness of ~0.8%.

For the optimized trajectory the velocity is very nearly a linear function of piston position and roughly exponential in time. This occurred for all cases that we examined. The fact that the velocity is higher at small X indicates the relatively greater importance of the heat leak when the gas temperature is high. The average velocity is much higher than with the conventional motion. This results in a considerable increase in the friction losses on this stroke, but reduces the friction losses on the nonpower strokes as well as producing a substantial reduction in the heat leak losses. The numerical results are summarized in Tables II and III.

For the limited acceleration cases a_{max} was varied from 5×10^3 to 5×10^4 m sec^{-2}. The lower of these values corresponds roughly to the maximum acceleration in the conventional motion. With the higher acceleration results very similar to the unconstrained case were obtained.

In Fig. 3 we plot the optimized power strokes for maximum accelerations of 5×10^3 and 5×10^4 m sec^{-2} as well as for the unconstrained case. Note that the latter two are virtually identical, except at the ends of the stroke. The lower acceleration case is also quite similar to the other two in the interior segment. However, the velocity for this example is distinctly higher, apparently in order to compensate for time lost on the other segments.

Figure 4 shows the conventional motion, the lowest acceleration case, and the "symmetric" case. This last case is one for which the trajectory was required to be the same on all four strokes. The maximum acceleration was 2×10^4 m sec^{-1}. Note that this curve is more level than the other two, this is to hold friction losses down. The peak velocity is lower than that for the conventional motion, but much closer to the beginning of the stroke so as to minimize heat leak losses in this region. The low acceleration case is similar to the conventional motion but with a higher peak. This increases friction losses on this stroke, but reduces losses on the other strokes as well as those due to heat leak.

The velocity as a function of time for the whole cycle is shown in Fig. 5. This is for the case $a_{max} = 2\times10^4$ cm sec^{-2}.

B. Comparison of optimal and conventional engines

The results of the calculations are summarized in Tables III–V. Table III contains the results for the unlimited

TABLE IV. Results for limited acceleration.

a_{max} msec^{-2}	v_{max} msec^{-1}	t' msec	W_P J	W_T J	W_f J	W_Q J	Q J	W_Q/Q	T_f K	ϵ
conv.	13.6	8.33	503	276	67	43	224	0.19	1095	0.633
5×10^3	17.1	7.62	500	279	63	43	210	0.20	1130	0.640
1×10^4	18.6	6.63	508	293	58	34	186	0.18	1160	0.672
2×10^4	20.5	6.07	513	300	58	28	172	0.16	1180	0.689
5×10^4	22.4	5.90	516	304	57	24	167	0.14	1185	0.698
unconstrained	25.4	5.48	518	307	58	21	156	0.13	1200	0.705
symmetric	13.1	8.32	511	290	57	38	220	0.17	1090	0.666

acceleration calculations, Table IV has those for limited acceleration, and Table V shows the percent improvements in effectiveness, heat leak losses, and friction losses. The various cases refer to different choices of friction coefficient, heat leak coefficient, and cycle time. Case I refers to the conditions listed in Table I. The parameters used for the different cases are listed in Table II.

From Tables III and IV we note several interesting effects. First, optimization has a much more pronounced effect on W_Q than on Q. This is due to the fact that the reduction in the heat leak is achieved primarily at the beginning of the stroke. Due to the higher temperature the heat saved here will be used with a greater efficiency than that saved later in the stroke. Also, as should be expected, larger values of Q are associated with smaller values of T_f. Finally, increasing or decreasing the total cycle time (cases IV and V) causes similar, but relatively smaller, changes in t'.

From Table V we can see that the improvement in effectiveness is primarily due to the reduction of heat leak losses. For the unlimited acceleration cases the effectiveness is improved by between 9.8% and 17.6%. The largest improvements are for those cases where the heat leak is most important. Whereas W_Q is reduced by between 42% and 71%, the friction losses are never reduced by more than 20%. For the two cases, II and V, with the largest heat leak losses the friction losses actually increase.

The most striking thing about the results of the limited acceleration calculations is that significant improvements in effectiveness can be achieved with moderate accelerations. An upper limit to acceleration of only 1×10^4 m sec^{-2} (slightly less than the maximum for the conventional motion at 4800 rpm) produces an improvement of 6.2%. With an acceleration of 5×10^4 m sec^{-2} some 90% of the maximum improvement is achieved.

Also from Table V we see that for $a_{max} > 1\times10^4$ m sec^{-1} there is little reduction in friction losses. Beyond this point the improvement in effectiveness is virtually entirely due to the reduction of heat leak losses. This again illustrates the extreme importance of the heat leak on the initial portion of the power stroke.

Finally, we consider the "symmetric" case. A linkage which produces the same motion on each stroke would certainly be easier to design than one that produces different piston motions. From Tables IV and V we see that with this constraint we can still improve the effectiveness by 5.1%. This is about 60% of the improvement realized without this constraint.

The result that the main portion of the losses are due to reducing the heat leak does not mean that the frictional effects are less important. The presence of friction determines the extent to which the heat leak can be reduced. If no friction were present the unlimited acceleration case would permit the heat leak to be eliminated altogether. The two loss terms are not independent and must both be included in order to get a reasonable result.

TABLE V. Improvements due to optimization.

Case	a_{max} m sec^{-2}	% increase in ϵ	% decrease in W_Q	% decrease in W_f
I	5×10^3	1.1	−0.14	5
I	1×10^4	6.2	21	12
I	2×10^4 [a]	5.1	12	14
I	2×10^4	8.7	34	14
I	5×10^4	10.1	44	14
I	∞	11.2	53	13
II	∞	17.6	71	−10.5
III	∞	9.8	61	20
IV	∞	11.4	42	18
V	∞	14.7	66	−1.3

[a]Symmetrical case.

V. CONCLUSIONS

In summary, we have found that, with our model, optimizing the piston motion has the potential of improving internal combustion engine efficiency by more than 10%. This is primarily due to reduction of the heat leak losses on the initial portion of the power stroke. Even with fairly strong constraints on the piston motion significant improvements can be made. These results may turn out to be either optimistic or conservative for a real engine. The results suggest that a more intense investigation of this means of improving efficiency is warranted.

In addition to improving the effectiveness and power there are three other advantages that may result from reducing friction and heat load on the engine. First, a lengthening of the engine life may result. Second, the cooling demands are greatly reduced, perhaps enough to make air cooling feasible. Finally, the fact that the exhaust gas temperature is higher for the optimized engine may be advantageous for the operation of catalytic converters. These factors may make it possible to make further changes in engine operating conditions, which, in turn, could produce further increases in efficiency or reduce emissions.

The development of an alternative mechanism for connecting the piston to the crankshaft has advantages in addition to those discussed here. Improved linkages could produce reductions in the shearing stress of the piston against the cylinder wall. This would increase both efficiency and engine lifetime. Another possibility is in the development of the variable displacement engine.[12] Such an engine has considerable potential for improving efficiency and would also require new linkage mechanisms.

It is possible that the maximum allowable acceleration may be dictated by thermodynamic, rather than mechanical, considerations. By this we refer to considerations of exhaust blowdown and time loss. By optimizing the piston motion during combustion it may be possible to reduce both time loss and emissions.

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APPENDIX: OPTIMAL CONTROL THEORY

Optimal control theory differs from the classical calculus of variations in two ways. The first is that optimal control theory gives the equations of motion in Hamiltonian rather than Lagrangian form. The second and more important difference is that optimal control theory can deal with inequality constraints.

The general optimal control problem is stated as follows. We wish to maximize the functional

$$J(\mathbf{v},\mathbf{y}) = \int_{t_0}^{t_1} F[\mathbf{y}(t),\mathbf{v}(t)]\, dt$$

subject to the constraints

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y},\mathbf{v}),$$

and

$$\mathbf{R}(\mathbf{v}) \leqslant \mathbf{0},$$

and the initial conditions

$$\mathbf{y}(t_0) = \mathbf{y}_0.$$

The components of $\mathbf{v}$ are called the control variables and the components of $\mathbf{y}$ are the state variables. The Hamiltonian is defined by

$$H = F + \boldsymbol{\lambda}\cdot\mathbf{f}.$$

Three conditions must be satisfied by the optimal path. First, the necessary conditions for an extremal path give the canonical equations of motion:

$$\dot{\mathbf{y}} = \frac{\partial H}{\partial \boldsymbol{\lambda}} = \mathbf{f}(\mathbf{y},\mathbf{v})$$

and

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{v}}.$$

That we obtain a maximum of J is assured by applying the *maximum principle*, that for every point $(\boldsymbol{\lambda},\mathbf{y})$ on the optimal path the optimal control vector $\mathbf{v}^*$ must satisfy the inequality

$$H(\mathbf{y},\mathbf{v}^*,\boldsymbol{\lambda}) \geqslant H(\mathbf{y},\mathbf{v},\boldsymbol{\lambda}),$$

where $\mathbf{v}$ is any control vector permitted by the inequality constraints.

Finally, if end-point condition $\mathbf{y}(t_1)$ is not given we must require that

$$\boldsymbol{\lambda}(t_1) = \mathbf{0}.$$

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